AFCRC-TN-58-239
ASTIA DOCUMENT No. AD 152473

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. MH-9

The Structure of Isotropic Turbulence at Very High Reynolds Numbers

ROBERT H. KRAICHNAN

CONTRACT No. AF19(604)2138
FEBRUARY, 1958

AFCRC-TN-58-239
ASTIA Document No. AD 152473

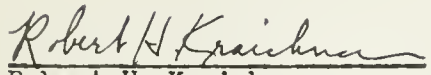
NEW YORK UNIVERSITY

Institute of Mathematical Sciences
Division of Electromagnetic Research

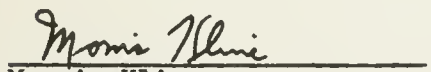
Research Report No. MH-9

THE STRUCTURE OF ISOTROPIC TURBULENCE AT VERY HIGH REYNOLDS NUMBERS

Robert N. Kraichnan



Robert H. Kraichnan



Morris Kline
Project Director

February, 1958

The research reported in this article has been sponsored by the Geophysics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, under Contract No. AF 19(604)2138.

New York, 1958

ABSTRACT

A recent, anticipatedly exact, analytical theory of homogeneous turbulence is specialized to stationary isotropic turbulence of very high Reynolds number, maintained by idealized, low wavenumber isotropic driving forces. The characteristic wavenumber $k_o = \varepsilon/v_o^3$ and Reynolds number $R_o = v_o k_o^{-1}/\nu$, where v_o is the rms velocity in any direction, ε is the power dissipated per unit mass, and ν is the kinematic viscosity, are introduced. For $R_o^{1/3} \gg 1$ it is found that the inertial and dissipation ranges extend over wavenumbers satisfying $k_o \ll k \ll R_o k_o$. In this case, the time correlation function and ensemble-averaged impulse response function for an inertial or dissipation range Fourier mode are found to be $J_1(2v_o k\tau)/(v_o k\tau)$ and $\exp(-\nu k^2\tau)J_1(2v_o k\tau)/(v_o k\tau)$, respectively. The energy spectrum in these ranges is determined by a nonlinear integral equation, involving the time correlation and response functions, which is suitable for solution by iteration. The solution is of the form $E(k)/v_o\nu = (k/k_d)^{-3/2}f(k/k_d)$, where $k_d = R_o^{2/3}k_o$ and f is a universal function. In the inertial range it reduces asymptotically to $E(k) = f(0)(\varepsilon v_o)^{1/2}k^{-3/2}$. The parameter f(0) can be obtained by quadratures, without solving the integral equation for E(k). Spectral energy transport throughout the inertial and dissipation ranges, and also up through wavenumbers $k \gtrsim R_o k_o$, is found to proceed by a cascade process essentially local in wavenumber space; the direct power delivered by all modes below k to all modes above $k' \gg k$ is of order $\varepsilon(k/k')^{3/2}$ if k and k' both lie within the inertial range. The spectrum contains an exponential factor for $k \gg k_d$, and mean-square velocity derivatives of all orders are finite. For $R_o^{1/3} \gg 1$ the skewness factor of the distribution of the nth-order longitudinal velocity derivative approaches the asymptotic form $A_n R_o^{-1/6}$, where A_n is a universal constant.

The $k^{-3/2}$ inertial range law, and not the Kolmogorov $k^{-5/3}$ law, is found to be followed closely by the one-dimensional spectrum of the total kinetic energy measured by Laufer on the axis of fully turbulent pipe flow at Reynolds number $R_o^{1/3} \sim 15$. The nature of the isotropic solution described above suggests that the solution for anisotropic turbulence may yield anisotropic asymptotic partition of energy for wavenumbers in the inertial and dissipation ranges because of anisotropy of the time correlation and response functions for $k \lesssim R_o k_o$. This offers a possibility of explaining the high wavenumber anisotropy found by Laufer. It is found that the asymptotic dissipation range scaling given by the present theory is better supported by grid turbulence results of Stewart and Townsend ($R_o^{1/3} \sim 2.3\text{-}3.7$) than the Kolmogorov scaling.

It is stressed that extreme caution must be exercised in interpreting the apparently favorable experimental evidence as support for the theory.

A comparison is given between the present theory, the Kolmogorov theory, Heisenberg's eddy-viscosity theory, and the analytical theories of Heisenberg and Chandrasekhar. The origin of the disagreement with Kolmogorov's conclusions is that on the present theory the energy containing modes do not have the trivial convective action of a uniform velocity field on the high k dynamics. Instead, they mix weakly statistically interdependent high k modes into each other, thereby importantly affecting the triple phase correlations which determine the mean energy transport within the high k region. The theories of Heisenberg and Chandrasekhar based on assuming zero fourth-order cumulants may be obtained from the present theory by discarding some of the dynamical coupling terms in the statistical equations of motion and ignoring the effect of the dynamical coupling on the modal response functions. These theories also are inconsistent with the Kolmogorov theory. A modification of Heisenberg's heuristic theory is examined

which involves a rational expression for the eddy viscosity and leads to a $k^{-3/2}$ inertial range law. The energy transport expression given by the eddy-viscosity theory (modified or not) is found in several respects to be less plausible physically than that of the present theory. A brief comparison is given with the theories of Proudman and Reid, and Tatsumi. It is concluded that these theories do not yield an inertial range.

TABLE OF CONTENTS

1. INTRODUCTION

A theory recently has been proposed (Kraichnan, 1958, hereinafter cited as Paper I) which is anticipated to provide an exact and (in principle, at least,) complete statistical description of turbulence that is homogeneous in at least one direction. The theory was explicitly worked out in Paper I only for the analytically simplest case of stationary, completely homogeneous turbulence, for which stationary homogeneous driving forces must be invoked. In the present paper, a further specialization is made to stationary isotropic turbulence. The theory is employed to yield the spectral structure, in both frequency and wavenumber, of the inertial and dissipation ranges for very high Reynolds numbers. These asymptotic results are compared with experiment, and the theory itself is examined in relation to current theories. An attempt is made to bring out as fully as is practicable the physical significance of the basic hypotheses, analytical structure, and predictions of the theory. Reference should be made to Paper I for the fundamental statistical mechanical aspects of the theory and for the derivation of the basic statistical equations of motion which form the starting point of the present investigation.

2. FUNDAMENTALS OF THE THEORY

2.1. The weak dependence hypothesis

We may start by considering a uniform incompressible fluid confined in a cubical box of side L. The velocity field $\tilde{u}_i(\underset{\sim}{x},t)$ within the box may be expanded in the usual fashion in a spatial Fourier series

$$\tilde{u}_i(\underset{\sim}{x},t) = \sum_{\underset{\sim}{k}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} u_i(\underset{\sim}{k},t), \tag{2.1}$$

and the coefficients $\underset{\sim}{u}(\underset{\sim}{k},t)$ may be considered the dynamical degrees of freedom of the fluid. In terms of them, the Navier-Stokes equation may be written

$$\dot{u}_i(\underset{\sim}{k},t) + \nu k^2 u_i(\underset{\sim}{k},t) = - ik_m P_{ij}(\underset{\sim}{k}) \sum_{\underset{\sim}{p}+\underset{\sim}{q}=\underset{\sim}{k}} u_j(\underset{\sim}{p},t)u_m(\underset{\sim}{q},t), \tag{2.2}$$

where the dot indicates time differentiation and the operator

$$P_{ij}(\underset{\sim}{k}) = \delta_{ij} - k^{-2}k_i k_j$$

expresses the action of the pressure forces in maintaining the incompressibility property

$$k_i u_i(\underset{\sim}{k},t) = 0. \tag{2.3}$$

The kinematic viscosity is denoted by ν. The summation in (2.2) runs over all wavevectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ allowed by the boundary conditions, so that each Fourier mode is coupled to essentially every other mode.

Let us assume a state of turbulent motion such that away from the walls the flow is statistically homogeneous in the sense that properties defined as averages over sufficiently large volumes do not vary with location of the volumes in the box. This implies that L must be much larger than characteristic length scales of the turbulence; for exact homogeneity, the effects of the walls must be vanishingly small, and we must have L infinite. Let us then imagine a sequence of boxes of increasing L. As we go to larger and larger members of the sequence, the spacing between adjacent wavevectors allowed by the boundary conditions decreases; the density of allowed modes in k space is $\propto L^3$. For L infinitely large we may expect that an infinitely large number of $\underset{\sim}{u}(\underset{\sim}{k},t)$ are excited in any given wavevector range, no matter how small. This is true regardless of how small may be the Reynolds number of the turbulence based on the rms turbulent

velocity and the turbulent macroscale. (We may keep these quantities constant as $L \to \infty$).

As $L \to \infty$, each $\underset{\sim}{u}(\underset{\sim}{k},t)$ is determined by contributions from more and more individual x space volumes of any given size. Interference effects of all the individual volumes may be expected to give rise to increasingly rapid and random oscillations of phase and amplitude as a function of $\underset{\sim}{k}$. On the other hand, since any neighborhood in k space, no matter how small, contains an infinite number of excited degrees of freedom, as $L \to \infty$, we may expect that averages over all the $\underset{\sim}{u}(\underset{\sim}{k},t)$ in any such small neighborhoods will yield smooth results independent of the precise sizes or shapes of the neighborhoods. In fact, averaging over tiny neighborhoods in k space is equivalent to averaging over large regions in x space. The averaging over tiny neighborhoods may be replaced by averaging over a suitable ensemble of flows. This latter kind of statistical average will be understood hereinafter; it leads more elegantly to identical results.

As we take the limit $L \to \infty$, the number of bilinear terms on the right side of (2.2) increases without limit, but any given wavevector appears in at most two terms in the sum (once as $\underset{\sim}{p}$ and once as $\underset{\sim}{q}$). Therefore it would appear that, in the limit, the dynamical coupling among any finite number of modes is infinitely weak, since, in the equation of motion of each member of the group, the others appear in only a finite number of the infinitely many bilinear terms which contribute.

It seems natural to assume that turbulence should be described by a statistical distribution of the velocity field which is as random, in some suitable sense, as the equations of motion and the constraints on the system permit. In terms of the k space representation, we may express this assumption

as requiring that the various $\underset{\sim}{u}(\underset{\sim}{k},t)$ be as statistically independent as is permitted by the equations of motion and constraints. The above observation about the weakness of dynamical coupling within finite groups of modes then suggests the following explicit hypothesis about the statistical structure of homogeneous turbulence:

All statistical moments formed from any finite number of the Fourier modes have normalized values which tend in the limit $L \to \infty$ to values corresponding to some distribution in which the Fourier modes are completely statistically independent, subject to (2.3) and the reality condition $\underset{\sim}{u}^*(\underset{\sim}{k},t) = \underset{\sim}{u}(-\underset{\sim}{k},t)$. We shall call this the weak dependence hypothesis. It is important for the consistency of the hypothesis that all other modes are paired differently in the bilinear terms in the equations of motion (2.2) for different given modes. If this were not so, strong statistical dependencies might arise despite the weak dynamical coupling of the given modes.

It should be noted carefully that weak dependence, which can be defined only in terms of a limiting process, is very different from independence of the Fourier modes, certain properties of which have been assumed by Millionshtchikov (1941), Heisenberg (1948), Proudman and Reid (1954), Chandrasekhar (1955), Tatsumi (1957), and other authors. It is known experimentally that the joint probability distribution of the velocity field at more than one point is not normal (Batchelor, 1953). In particular, the skewness and flatness factors of the two-point distribution do not have normal values. This implies that certain triple moments of the $\underset{\sim}{u}(\underset{\sim}{k},t)$ cannot vanish as they would in an independent distribution and that certain fourth-order moments are not related to second-order moments as in an independent distribution. As a complement to the experimental evidence, it readily is seen from (2.2) that non-zero mean energy transfer

among the $\underset{\sim}{u}(\underset{\sim}{k},t)$ requires certain non-vanishing triple moments, and it has been shown also that the equations of motion imply non-independent relationship of fourth-order and second-order moments (Kraichnan, 1957). However, there is no conflict between the experimental evidence and the hypothesis of weak dependence. We may illustrate this by examining the skewness factor $\langle(\partial\tilde{u}_1/\partial x_1)^3\rangle/\langle(\partial\tilde{u}_1/\partial x_1)^2\rangle^{3/2}$. If $\langle(\partial\tilde{u}_1/\partial x_1)^3\rangle$ is expanded as a summation over triple moments of the Fourier modes, it readily is seen that the number of terms in the sum which involve only modes with wavenumbers below any arbitrary value increases as L^6 as $L\to\infty$. The fact that the individual triple moments approach zero as the limit is taken by no means implies that the sum approaches zero. In a similar fashion it may be seen that the property of weak dependence does not imply normal flatness factor for the distribution of $\partial\tilde{u}_1/\partial x_1$, or normal relations among other moments of the x space distribution. Because of the mutual interference effects from all parts of the box, mentioned above, weak dependence is consistent even with the sharply defined x space structures (shear fronts and vortex filaments) which make up very high Reynolds number homogeneous turbulence.

2.2. The direct interaction hypothesis

It is clear from (2.2) that the total nonlinear interaction of the Fourier modes is the resultant of elementary interactions each involving a triad of modes. As $L\to\infty$, the number of contributing triad interactions on the right of (2.2) increases as L^3. The hypothesis of weak dependence suggests an approach to the problem of turbulence dynamics in which each triad coupling is treated effectively as an infinitesimal perturbation on the total motion due to all the triad couplings, and the viscous forces, acting together. Such an approach can be carried out in an apparently self-consistent manner and used to evaluate the physically important triple correlations of the $\underset{\sim}{u}(\underset{\sim}{k},t)$, provided a second, related statistical hypothesis is made.

Let $\underset{\sim}{k}$, $\underset{\sim}{p}$, $\underset{\sim}{q}$ be three wavevectors such that $\underset{\sim}{k} - \underset{\sim}{p} - \underset{\sim}{q} = 0$. The dynamical interaction of $\underset{\sim}{u}(\pm\underset{\sim}{k})$, $\underset{\sim}{u}(\pm\underset{\sim}{p})$, $\underset{\sim}{u}(\pm\underset{\sim}{q})$ can take place along a direct path and along indirect paths. By the direct interaction we mean that given by the terms bilinear in components of $\underset{\sim}{u}(\pm\underset{\sim}{p})$, $\underset{\sim}{u}(\pm\underset{\sim}{q})$ in the equation of motion (2.2) for $\underset{\sim}{u}(\pm\underset{\sim}{k})$, and the corresponding terms in the equations of motion of $\underset{\sim}{u}(\pm\underset{\sim}{p})$ and $\underset{\sim}{u}(\pm\underset{\sim}{q})$; that is, the bilinear terms in the equations of motion of these amplitudes which do not involve amplitudes associated with any wavevectors other than $\pm\underset{\sim}{k}$, $\pm\underset{\sim}{p}$, $\pm\underset{\sim}{q}$. By the indirect interaction we mean the dynamical coupling among these three modes which would remain if the terms representing their direct interaction were removed from the equations of motion. The indirect coupling exists by virtue of the coupling of each of the given three modes, through the remaining bilinear terms, to all the other modes in the system.

Let us now consider the triple correlation $\langle u_i^*(\underset{\sim}{k},t)u_j(\underset{\sim}{p},t')u_m(\underset{\sim}{q},t'')\rangle$. It is clear that the value of this correlation is affected by the presence of the direct interaction, for this interaction produces increments in $u_i^*(\underset{\sim}{k},t)$ which bear a definite phase relation to products of the form $u_j(\underset{\sim}{p},t)u_m(\underset{\sim}{q},t)$. We shall make the hypothesis that as the limit $L\to\infty$ is approached the triple correlation becomes determined by the direct interaction in the sense that if the direct interaction terms were not present the value of the triple correlation would be small without limit compared to its value with these terms present. We shall call this the direct interaction hypothesis.

Several arguments may be presented in favor of this hypothesis, and there exists analytical support of an a posteriori nature, as discussed in Paper I. A rigorous verification has not been obtained. For present purposes we shall simply assert the plausibility of the assumption that the coupling of three degrees of freedom by the indirect interaction alone fails to convey any phase information among them in the limit where an infinite number of weakly

statistically dependent modes constitute the system-as-a-whole through which the indirect interaction takes place. This seems intuitively reasonable if one views turbulence as a mixing or randomizing process. In any event, we do not present the hypothesis as an approximation. We consider both the weak dependence and the direct interaction hypotheses to express exact properties of exactly homogeneous turbulence ($L \to \infty$). There does not appear to be any contrary evidence. The extension of the direct interaction hypothesis to certain fourth-order moments and to third-order moments involving possible external forces driving the turbulence is given in Paper I.

2.3. Perturbation theory determination of triple moments

We may evaluate triple moments of the form $\langle u_i^*(\underset{\sim}{k},t)u_j(\underset{\sim}{p},t')u_m(\underset{\sim}{q},t'')\rangle$ on the basis of the direct interaction hypothesis by first considering a hypothetical system which differs from the actual system in that the several terms giving the direct interaction are missing. For this system we assume the triple moment is negligible. Then we obtain the value of the triple moment in the actual system by switching on the direct interaction terms as a perturbation. It is clear from what has been said that these few terms represent an infinitesimal perturbation for $L \to \infty$ and that, consequently, first-order perturbation theory may be used without error. Thus, in the limit $L \to \infty$ we have

$$\langle u_i^*(\underset{\sim}{k},t)u_j(\underset{\sim}{p},t')u_m(\underset{\sim}{q},t'')\rangle = \langle \delta u_i^*(\underset{\sim}{k},t)u_j(\underset{\sim}{p},t')u_m(\underset{\sim}{q},t'')\rangle + \langle u_i^*(\underset{\sim}{k},t)\delta u_j(\underset{\sim}{p},t')u_m(\underset{\sim}{q},t'')\rangle + \langle u_i^*(\underset{\sim}{k},t)u_j(\underset{\sim}{p},t')\delta u_m(\underset{\sim}{q},t'')\rangle \,, \tag{2.4}$$

where $\delta u_i(\underset{\sim}{k},t)$ is the infinitesimal perturbation induced in $u_i(\underset{\sim}{k},t)$.

In general, the linear response to an arbitrary solenoidal infinitesimal perturbation term $\xi_i(\underset{\sim}{k},t)$ on the right side of (2.2), switched on at $t = 0$, is

$$\delta u_i(\underset{\sim}{k},t) = \int_o^t \zeta_{ij}(\underset{\sim}{k};t,s)\xi_j(\underset{\sim}{k},s)ds, \tag{2.5}$$

where $\zeta_{ij}(\underset{\sim}{k};t,s)$ is the infinitesimal impulse response tensor. Since the system is nonlinear, $\zeta_{ij}(\underset{\sim}{k};t,s)$ is some (so far undetermined) functional of the values of all the Fourier amplitudes. If now $\underset{\sim}{\xi}$ represents the direct interaction terms and appeal is made to the weak dependence hypothesis, the ensemble averages on the right side of (2.4) may be reduced to expressions involving only the ensemble averages $\langle\zeta_{ij}(\underset{\sim}{k};t,s)\rangle$, . . . of the impulse response tensor and covariances of the form $\langle u_i(\underset{\sim}{k},t)u_n^*(\underset{\sim}{k},s)\rangle$, ... To complete the evaluation of the triple moments it remains to determine the averaged impulse response tensor. Equations of motion for this tensor may be obtained by considering further the response of the system to arbitrary external perturbations. These equations and the equations of motion for the covariance tensor serve to determine the two tensors simultaneously. The complete derivation for the stationary case is given in Paper I.

Using an extension of the direct interaction hypothesis, certain fourth-order moments of the Fourier amplitudes also may be expressed in terms of the covariance tensor and the averaged impulse response tensor, and a corresponding reduction may be made for certain third-order moments involving random external forces which may be driving the system. It is shown in Paper I that from knowledge of these moments it is possible to obtain the complete set of higher-order moments of a distribution which satisfies the equations of motion exactly, and thus to obtain a complete statistical description of the turbulence. In the present paper, however, we shall be concerned only with second- and third-order moments.

3. FORMULATION FOR STATIONARY ISOTROPIC TURBULENCE

The theory described above takes its simplest form for the stationary isotropic case. The concept of stationary isotropic turbulence involves a considerable idealization of physical flows. For turbulence to be stationary there must be driving forces to replace energy lost by viscosity; in order to maintain isotropy the driving forces must themselves be isotropic, and this is physically unrealistic. (We may visualize such forces, perhaps, as due to a random volume distribution of stirring devices.) Despite the unrealistic nature of stationary isotropic turbulence, it is valuable for study. Analytically, it is the simplest case; physically, it is plausible, if the driving forces invoked are confined to low wavenumbers, that the structure at high wavenumbers will not differ appreciably from that of freely decaying isotropic turbulence (also an idealization). We shall return to this question after exploring the consequences of the stationary isotropic theory.

The stationary isotropic state may be described in a mathematically consistent fashion by replacing (2.2) with

$$(3.1) \qquad \dot{u}_i(\underset{\sim}{k},t) + \nu k^2 u_i(\underset{\sim}{k},t) = -\, ik_m P_{ij}(\underset{\sim}{k}) \sum_{\underset{\sim}{p}+\underset{\sim}{q}=\underset{\sim}{k}} u_j(\underset{\sim}{p},t)u_m(\underset{\sim}{q},t) + f_i(\underset{\sim}{k},t),$$

where $f_i(\underset{\sim}{k},t)$ is a statistically stationary and isotropic solenoidal forcing term. An alternate procedure, which in some ways is more closely related to the physical mechanism of support by a mean shear flow, is the introduction of negative viscosity at low wavenumbers rather than external forcing terms.

The conditions of stationarity and isotropy require that the covariances constructed from $\underset{\sim}{u}(\underset{\sim}{k},t)$ and $\underset{\sim}{f}(\underset{\sim}{k},t)$ have the forms

$$(3.2)\qquad (L/2\pi)^3\langle u_i(\underset{\sim}{k},t+\tau)u_j^*(\underset{\sim}{k}',t)\rangle = \tfrac{1}{2}\,\delta_{\underset{\sim}{k},\underset{\sim}{k}'}P_{ij}(\underset{\sim}{k})U(k,\tau),$$

$$(L/2\pi)^3\langle f_i(\underset{\sim}{k},t+\tau)f_j^*(\underset{\sim}{k}',t)\rangle = \tfrac{1}{2}\,\delta_{\underset{\sim}{k},\underset{\sim}{k}'}P_{ij}(\underset{\sim}{k})F(k,\tau),$$

$$(L/2\pi)^3\mathrm{Re}\langle f_i(\underset{\sim}{k},t+\tau)u_j^*(\underset{\sim}{k}',t)\rangle = \tfrac{1}{2}\,\delta_{\underset{\sim}{k},\underset{\sim}{k}'}P_{ij}(\underset{\sim}{k})G(k,\tau),$$

where the scalars U, F, and G are all real and U and F are even functions of τ. In the limit $L\to\infty$ (which is required for rigorous isotropy),

$$(3.3)\qquad U(k,\tau) = (2\pi)^{-3}\int \tilde{U}(x,\tau)e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}d^3x,$$

where $\tilde{U}(x,\tau) = \langle \tilde{u}_i(\underset{\sim}{x}+\underset{\sim}{y},\ t+\tau)\tilde{u}_i(\underset{\sim}{y},t)\rangle$, with corresponding expressions for $F(k,\tau)$ and $G(k,\tau)$.*

From (3.1) we may obtain the equation of motion

$$(3.4)\qquad \dot{U}(k,\tau) + \nu k^2 U(k,\tau) = S(k,\tau) + G(k,\tau),$$

where the dot now signifies τ differentiation and

$$(3.5)\qquad S(k,\tau) = (L/2\pi)^3\ \mathrm{Im}\Big\{k_m \sum_{\underset{\sim}{p}+\underset{\sim}{q}=\underset{\sim}{k}} \langle u_i(\underset{\sim}{p},t)u_m(\underset{\sim}{q},t)u_i^*(\underset{\sim}{k},t-\tau)\rangle\Big\}.$$

The perturbation procedure described in the preceding section may be used to reduce the triple moment on the right side of (3.5). Since the turbulence is stationary and isotropic, the ensemble average of the impulse response tensor must display these properties also. Thus, (2.5) leads to

---

* In accordance with the convention adopted in Paper I, the normalization in (3.2) corresponds to the requirement that the velocity obey cyclic boundary conditions on opposite faces of a cube of side L, not necessarily vanish. To avoid confusion with another notation, $f_i(\underset{\sim}{k},t)$ was denoted by $F_i(\underset{\sim}{k},t)$ in Paper I. $F(k,\tau)$ has the same meaning in both papers.

$$\bar{\delta}u_i(\underset{\sim}{k},t) = \int_o^t g(k,t-s)\xi_i(\underset{\sim}{k},s)ds, \tag{3.6}$$

where $\bar{\delta}$ denotes the ensemble averaged induced perturbation and the impulse response function $g(k,t-s)$ is defined by $P_{ij}(\underset{\sim}{k})g(k,t-s) = \langle \zeta_{ij}(\underset{\sim}{k};t,s)\rangle$. Then, as detailed in Paper I. the perturbation procedure yields, in the limit $L\to\infty$,

$$S(k,\tau) = \pi k\iint_\Delta pqdpdq\int_o^\infty ds[a(k,p,q)g(k,s)U(p,s+\tau)U(q,s+\tau) - b(k,p,q)g(p,s)U(k,s-\tau)U(q,s)] , \tag{3.7}$$

where the domain of integration extends over all wavenumbers p and q such that p,q, and k can form the legs of a triangle. The quantities a and b are defined by

$$a(k,p,q) = (1-xyz-2y^2z^2), \tag{3.8}$$

$$b(k,p,q) = 2k^{-1}p(-z+xy+x^2z+y^2z+2z^3+2xyz^2),$$

where x, y, z are the cosines of the interior angles opposite the legs k, p, q, respectively. They obey the identities

$$a(k,p,q) = a(k,q,p) \geq 0, \tag{3.9}$$

$$k^2b(k,p,q) = p^2b(p,k,q),$$

$$b(k,p,q)+b(k,q,p) = 2a(k,p,q),$$

which, as we shall see shortly, are connected with the energy conservation properties of the interaction.

Since the driving forces are arbitrary in any event, let us take them to be Gaussianly distributed. Then they may be regarded as sums of infinite numbers of infinitesimal statistically independent increments. Each increment individually may be considered an infinitesimal perturbation to the equations of motion; we thereby find

$$G(k,\tau) = \int_0^\infty g(k,s)F(k,s+\tau)ds. \tag{3.10}$$

By considering the effect of an arbitrary, statistically stationary and isotropic, solenoidal perturbation $\delta f_i(\underset{\sim}{k},t)$ on a statistical equation obtained by multiplying (3.1) through by $f_i^*(\underset{\sim}{k},t')$ and averaging, we may obtain the equation of motion

$$\dot{g}(k,\tau) + \nu k^2 g(k,\tau) = -\pi k \iint_\Delta pqdpdqb(k,p,q) \times \int_0^\tau g(k,\tau-s)g(p,s)U(q,s)ds \qquad (\tau \geq 0), \tag{3.11}$$

as described in Paper I. For $\tau < 0$, $g(k,\tau) = 0$, since there can be no response to a perturbation before it is applied. If $F(k,\tau)$ is prescribed, (3.4), (3.7), (3.10), and (3.11) form a complete system determining $U(k,\tau)$ and $g(k,\tau)$, provided suitable boundary conditions are imposed. The condition of stationarity requires

$$\dot{U}(k,0) = 0, \tag{3.12}$$

which may be formulated by (3.4) and (3.7) as an integral condition (4.2) expressing the balance of energy. The other required boundary condition is

$$g(k,+0) = 1 \tag{3.13}$$

which follows immediately from the definition of $g(k,\tau)$ and the fact that the equation of motion (3.1) is first-order.

4. THE SPECTRAL TRANSPORT OF ENERGY

The covariance scalars introduced in the last Section may be written

$$\tfrac{1}{2} U(k,\tau) = (4\pi k^2)^{-1} E(k) r(k,\tau), \tag{4.1}$$

$$F(k,\tau) = (4\pi k^2)^{-1} F(k) \mu(k,\tau),$$

where $E(k)$ is the kinetic energy spectrum and $F(k)$ the forcing spectrum, both per unit mass of fluid. The total kinetic energy per unit mass is $\int_o^\infty E(k)dk$. The quantity $r(k,\tau)$ is the time correlation function of the Fourier mode, normalized so that $r(k,0) = 1$, with a corresponding interpretation for $\mu(k,\tau)$. If we rewrite (3.4) in terms of the new quantities, set $\tau = 0$, and note the symmetry of the integration domain with respect to p and q, we obtain the relation

$$2\nu k^2 E(k) = \frac{1}{2}\iint_\Delta S(k|p,q)dpdq + F(k)\int_o^\infty g(k,s)\mu(k,s)ds, \tag{4.2}$$

where

$$S(k|p,q) = (k/pq)\{2k^2 a(k,p,q)\theta(k,p,q)E(p)E(q) - [p^2 b(k,p,q)\theta(p,k,q)E(q) + q^2 b(k,q,p)\theta(q,k,p)E(p)]E(k)\} \tag{4.3}$$

and

$$\theta(k,p,q) = \int_o^\infty g(k,s)r(p,s)r(q,s)ds. \tag{4.4}$$

The left side of (4.2) is the mean power dissipated by viscosity, per unit mass and per unit wavenumber. The second term on the right may be interpreted as the mean power input by the driving forces, per unit mass and per unit wavenumber. (Hereafter, all powers will be understood to refer to unit mass.) It should be noted that the linear dependence of this term on $F(k)$ is only apparent; the response function $g(k,s)$ is an implicit function of and μ determined by (3.4), (3.10), and (3.11). The first term on the right may be interpreted as a mean power input due to nonlinear interaction of the velocity modes. Thus, $S(k|p,q)dkdpdq$ represents the mean power delivered to the Fourier modes in the interval dk as a consequence of their triad interactions with all pairs of modes of which one member lies in each of the intervals dp and dq. $S(k|p,q)$ is symmetric in p and q, and the factor $\frac{1}{2}$ in (4.2) occurs because each pair of modes is counted twice in the integration.

It may be deduced readily from (3.1) that the interaction of any triad of Fourier modes is individually conservative; this implies

$$S(k|p,q) + S(p|q,k) + S(q|k,p) = 0 \tag{4.5}$$

which may also be obtained easily from (4.3) by using the trigonometric identities (3.9) and the fact that $\theta(k,p,q)$ is symmetric in p and q. The overall conservation property

$$\frac{1}{2}\int_0^\infty dk \iint_\Delta S(k|p,q)dpdq = 0 \tag{4.6}$$

follows directly from the detailed conservation relation.

It should be noted that while the concept of the elementary interaction of a <u>triad</u> of Fourier modes has a particular significance because of the detailed conservation property, the same is not true of the concept of exchange of energy between <u>pairs</u> of modes, which has been used frequently in turbulence theory. There is no way to tell how much of the energy a mode $\underset{\sim}{k}$ receives from interaction with modes $\underset{\sim}{p}$ and $\underset{\sim}{q}$ comes from $\underset{\sim}{p}$ rather than from $\underset{\sim}{q}$ (Paper I, footnote 7).

The quantities which enter expression (4.3) may be given fairly direct physical interpretations. We shall consider first the factors $\theta(k,p,q)$, $\theta(p,k,q)$, $\theta(q,k,p)$ which, from their dimensions, evidently represent characteristic times for the triad interaction. We may anticipate that these factors are essentially positive. The physical significance of the response function $g(k,s)$ is that it traces the relaxation of an initial applied excitation of mode k through the joint action of viscous decay and the energy exchange, or mixing action, with the infinitely large number of only weakly statistically inter-related other modes. We may expect that it falls to zero with increasing s in a fairly smooth fashion and that it does not exhibit strongly negative regions. The correlation function $r(k,s)$ also is determined by the general mixing action. We may expect

that it, too, is essentially positive and falls fairly smoothly to zero with increase of s. Thus $\Theta(k,p,q)$, $\Theta(p,k,q)$, and $\Theta(q,k,p)$ may be expected to be the order of characteristic correlation and relaxation times of the three modes involved.

The proportionality of the terms in $S(k|p,q)$ to the Θ factors may be interpreted as follows. A non-zero mean transfer of energy among the three modes requires that certain phase relations among these modes be established in a statistical sense. These relations are built up by the direct interaction of the three modes, according to the direct interaction hypothesis, but are simultaneously broken down by the viscous decay and effectively random mixing with all the other modes, as expressed by the g and r functions. Loosely speaking, we may say that $\Theta(k,p,q)$ is an effective time during which the triple phase correlation can build up before it is broken down.

The first of identities (3.9) indicates that $a(k,p,q)$ is never negative, and the third that $b(k,p,q)$ is typically positive. If, as anticipated, the Θ factors are positive, the terms in (4.3) involving a represents a positive flow of energy to mode k while those involving b represent a typically negative flow. The net flow is the resultant of these absorption and emission terms. It will be noticed that in contrast to the absorption term the emission terms are proportional to $E(k)$. This indicates that the energy exchange acts to maintain equilibrium. If the spectrum level were suddenly raised to much higher than the equilibrium value in a narrow neighborhood of k, the emission terms would be greatly increased while the absorption term would be little affected; thus, energy would be drained from the neighborhood and equilibrium re-established. The structure of the emission and absorption terms is such that in general we may expect the energy flow to be from strongly to weakly excited modes, in accord with general statistical mechanical principles.

Although, as was noted above, it is impossible to define unambiguously the transfer of energy between a pair of modes, it is meaningful to define a transport power $\Pi(k)$ as the mean power input to all modes of wavenumber higher than some value k from all modes of wavenumber less than k. For this purpose we divide all the triad interactions into four classes as shown in Figure 1. Classes A and B do not transfer energy across the boundary. $\Pi(k)$ is therefore the power input to all modes $k' > k$ by all interactions in class C less the power input to all modes $k' < k$ by all interactions in class D. Thus,

$$(4.7) \qquad \Pi(k) = \frac{1}{2}\int_k^{2k} dk' \iint_{\Delta'}^{(p,q<k)} S(k'|p,q)dpdq - \frac{1}{2}\int_o^k dk' \iint_{\Delta'}^{(p,q>k)} S(k'|p,q)dpdq.$$

The upper limit on the first integral sign is taken as 2k instead of ∞ because for $k' > 2k$ there are no p and q such that $p,q < k$ and k', p, q form a triangle.

It follows from the conservation properties that the interaction power-input density appearing in (4.2) is related to $\Pi(k)$ by

$$(4.8) \qquad \frac{1}{2}\iint_{\Delta} S(k|p,q)dpdq = -\frac{\partial \Pi(k)}{\partial k}$$

5. RESPONSE AND TIME CORRELATION FUNCTIONS FOR HIGH WAVENUMBERS

5.1. The response function

The equations of motion (3.4) and (3.11), with $S(k,\tau)$ and $G(k,\tau)$ given by (3.7) and (3.10), are of a type which does not seem to have been studied, and it may be anticipated that in general their solution will present severe difficulties. It is not too difficult, however, to develop a solution which is asymptotically valid for the inertial and dissipation wavenumber ranges in the limit of very high Reynolds number. The simplest starting point is the equation

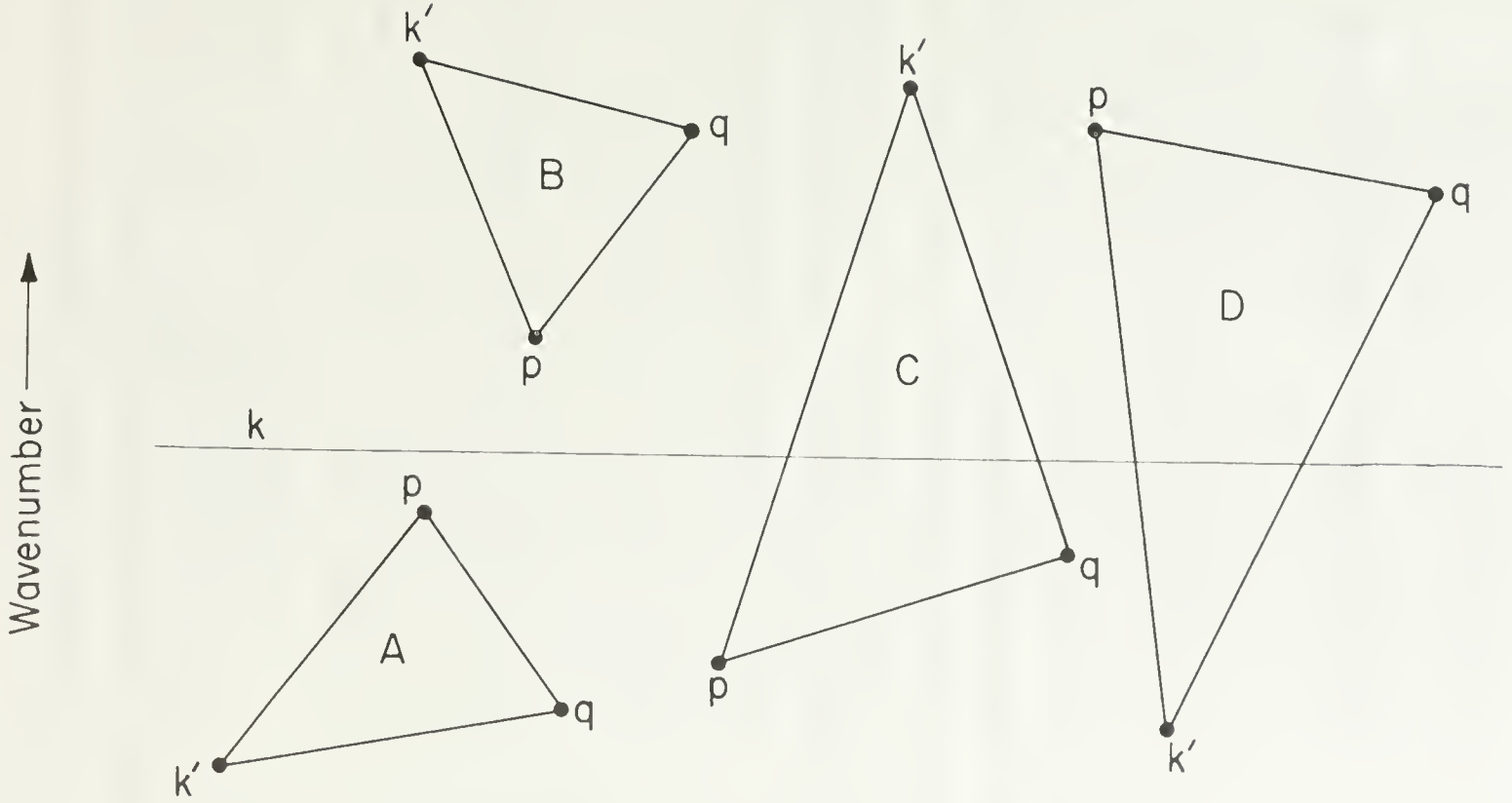


Figure 1. Classification of triad interactions for computing the transport power $\Pi(k)$.

of motion for $g(k,\tau)$.

Using (4.1) we may rewrite (3.11) in the form

$$(5.1)\qquad \dot{g}(k,\tau) + \nu k^2 g(k,\tau) = -\frac{k}{2}\iint_{\Delta} dpdq\,\frac{p}{q}b(k,p,q)E(q)\int_{o}^{\tau} g(k,\tau-s)g(p,s)r(q,s)ds \qquad (\tau \geq 0).$$

Now let us consider a k sufficiently high that all but a very small fraction of the total energy lies very far below k. Because of the factor $E(q)$, it appears plausible to assume that the right side of (5.1) should be dominated by contributions from the principal energy containing region $q \ll k$. This assumption is supported by direct inspection of the original equation of motion (3.1). It will be noticed that the coefficient $ik_m P_{ij}(\underset{\sim}{k})$ of the bilinear terms is independent of $|\underset{\sim}{p}|$ and $|\underset{\sim}{q}|$. Apart from certain angular dependence effects, the coupling of the mode $\underset{\sim}{k}$ to modes p or $q \ll k$ appears quite as strong intrinsically as its coupling to modes $\sim k$. Thus we might expect that the contribution of the bilinear terms to the motion should be dominated by terms involving the modes which contain most of the total excitation. Also, it is clear from the appearance of the factor k_m that, in general, the rate of change of $u_i(\underset{\sim}{k},t)$ ascribable to the nonlinear terms tends to increase with k. Thus we may anticipate that the characteristic times of the functions $g(k,s)$ and $r(k,s)$ decrease with increasing k.

Our assumptions are related to the observations in the x space representation that the time variation of the fine structure, viewed in a fixed coordinate system, should be dominated by the sweeping action of the energetic large scale motion and that small structures should be swept past fixed points more rapidly than large ones.

Returning to (5.1), we note that for $q \ll k$ the triangle relation gives $p \approx k$; then we may approximate $g(p,s)$ by $g(k,s)$, provided it is a reasonably smooth function. Since the time characterizing $r(q,s)$ should be very large compared to that of $g(k,s)$, we shall assume that in the region where $g(k,s)$ is significantly different from zero we may replace $r(q,s)$ by $r(q,0) = 1$. Finally, we may find from the definition (3.8) that, for $k \approx p$,

$$b(k,p,q) \approx 2\sin^2\beta,$$

where β is the interior angle between k and q. The wavenumber integration for q in the energy range now can be readily carried out, yielding

$$\iint_{\Delta} pq^{-1}b(k,p,q)E(q)dpdq \approx 2\int_o^{k_e} dqE(q)\int_{k-q}^{k+q} pq^{-1}\sin^2\beta dp \approx \frac{4}{3}k\int_o^{k_e} E(q)dq,$$

where $k_e \ll k$ is a wavenumber below which lies very nearly all the energy. Then, noting

$$\int_o^{k_e} E(q)dq \approx \int_o^{\infty} E(q)dq = \frac{3}{2}v_o^2,$$

where v_o is the rms velocity in any direction, we obtain the equation of motion

$$\dot{g}(k,\tau) + \nu k^2 g(k,\tau) = -v_o^2 k^2 \int_o^{\tau} g(k,\tau-s)g(k,s)ds. \qquad (\tau \geq 0). \tag{5.2}$$

If our assumptions and approximations are well-founded, this equation should be asymptotically exact for sufficiently high k.

The solution of the Laplace transform of (5.2) for the boundary condition (3.13) is a standard form, and we thereby find

$$g(k,\tau) = \exp(-\nu k^2\tau)J_1(2v_o k\tau)/(v_o k\tau) \qquad (\tau \geq 0). \tag{5.3}$$

A detailed investigation of the behavior of the right side of (5.1), with g given by (5.3) and r estimated from the results to be derived in Section 5.2, indicates the consistency of our assumptions and approximations. It is found that the restriction of the integration region to $q \ll k$ can give rise to appreciable fractional error in $g(k,\tau)$ only when $\tau \gg (\nu k^2)^{-1}$. For such τ, error can arise because the integral over s may be much larger for p, $q \sim \frac{1}{2}k$ than for $p \approx k$. However, $g(k,\tau)$ is very small for such τ, and the possible error is unimportant. Moreover, it decreases with increasing separation of k and the energy containing region. Our further approximations are found to lead to no difficulties.

The factor $\exp(-\nu k^2 \tau)$ in (5.3) is the response function appropriate to the decay of an applied excitation by viscosity without interaction with the other modes. The factor $J_1(2v_o k\tau)/(v_o k\tau)$ (plotted in Figure 2) represents the relaxation associated with the energy exchange between the given mode and others with very nearly equal wavevectors, due to the mixing action of the energy containing region. By this process, the applied excitation is mixed, or diffused, into many neighboring (but statistically weakly dependent) modes. The characteristic time $(v_o k)^{-1}$ associated with the mixing contribution to $g(k,\tau)$ is the same order as that which would be expected from the convection of a periodic pattern of wavenumber k by a uniform velocity of magnitude v_o. However, the action of the low wavenumbers in the present theory is very different from the convective action of a uniform field. Such a field does not produce mixing of neighboring modes and therefore gives rise to entirely coherent time dependence, rather than relaxation effects. This matter will be discussed further in Section 9.1.

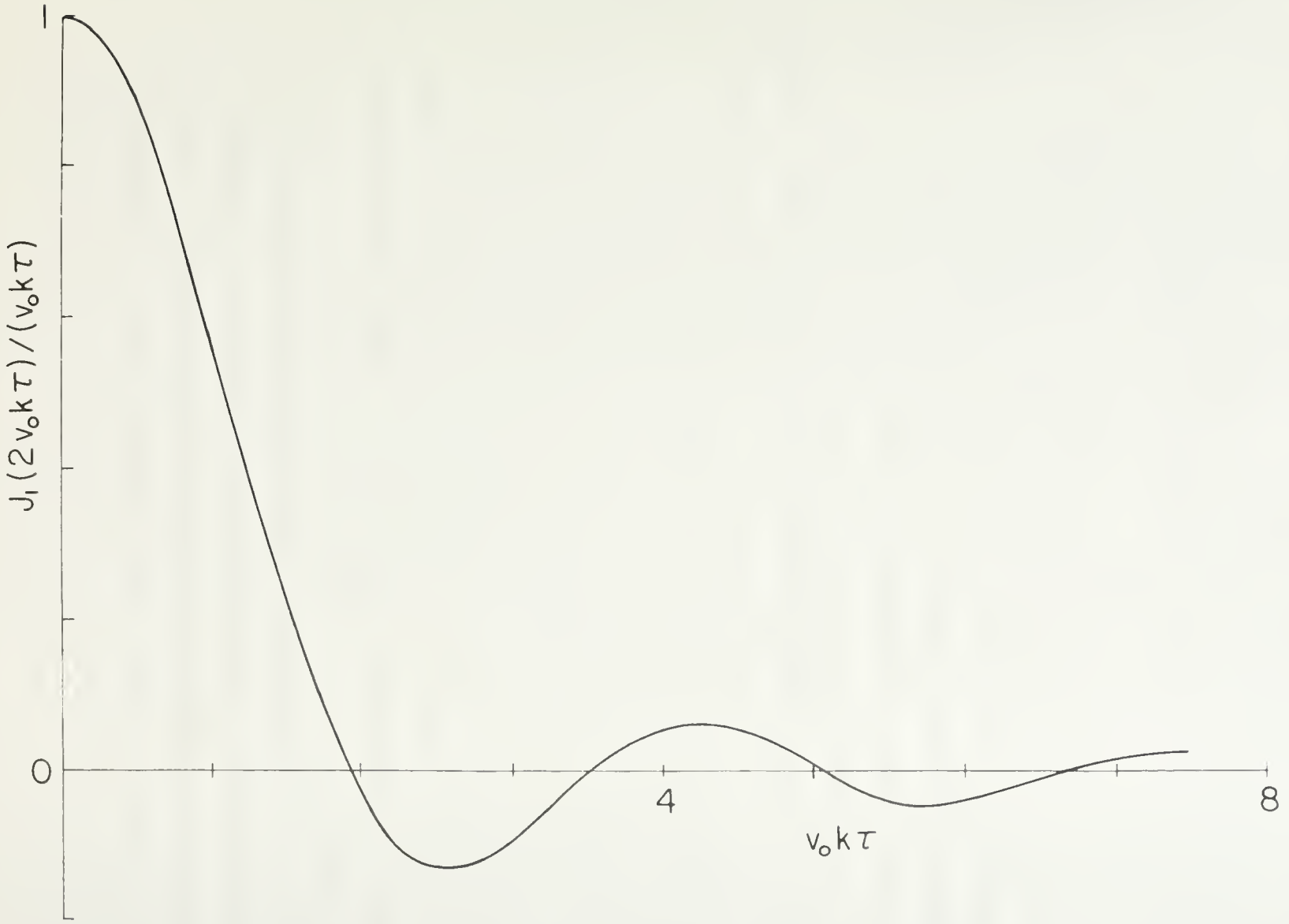


Figure 2. The response and time-correlation function $J_1(2v_0k\tau)/(v_0k\tau)$.

In accord with earlier anticipations, the response function given by (5.3) falls off fairly smoothly with increase of τ. However, there are oscillations, as shown in Figure 2. Although these are small, and therefore do not have much affect on the Θ factors and the magnitude of energy transport, they have a considerable intrinsic interest. In the language of linear systems, they indicate that the mixing with neighboring modes presents a partly reactive impedance to a given mode. This may be understood further in connection with a physical interpretation of the structure of the right side of (5.1). The transfer of part of an initially applied excitation from mode k to mode p involves two steps. First, a perturbation is induced in mode p. This is determined by the response function $g(p,s)$, and involves also the amplitude of a third mode, q, in accord with the nonlinear structure of (3.1). Second, the perturbation in mode p reacts upon mode k producing in it a change determined by the response function $g(k,\tau-s)$, and again involving the amplitude of mode q. The role of mode q in the two steps accounts for the expression $E(q)r(q,s)$. It is clear that for part of the energy of the initial excitation to be removed from mode k the reaction-induced perturbation must be effectively of opposite phase from the initial disturbance. The oscillations in g indicate that at certain difference-times the counter-perturbation actually "overshoots" so that there is in effect a small coherent oscillation of the initial excitation energy between modes k and p.

It should be noted that $g(k,\tau)$ plays a double role. It gives the (average) response to individual perturbations, but in doing so it must take account of the dynamical effect of an infinite number of repetitions of individual pairs of actions and reactions as just described. This is reflected in the

inhomogeneity of (5.1) and (5.2) in g. The right sides may be expanded by iteration into forms involving all powers of g, and many-fold time integrals. This reason for the nonlinearity of (5.1) (which is of higher degree than that of (3.1)) would exist even if the equations of motion of the system were linear.

5.2. The time correlation function

Measurements on real turbulent flows seem to indicate that the supply of energy from the mean flow, or from whatever other external source may be acting, is effectively confined to wavenumbers in and near the energy containing region. Then we may rewrite (3.4), with $S(k,\tau)$ given by (3.7), in the form

$$(5.4)\qquad \dot{r}(k,\tau) + \nu k^2 r(k,\tau) = \frac{k}{2}\iint_{\Delta}\left[k^2 a(k,p,q)\frac{E(p)E(q)}{E(k)}\int_{\tau}^{\infty} g(k,s-\tau)r(p,s)r(q,s)ds \right.$$

$$\left. -p^2 b(k,p,q)E(q)\int_{o}^{\infty} g(p,s)r(k,\tau-s)r(q,s)ds\right]\frac{dpdq}{pq}\ .$$

(In obtaining this form a slight change of time variable has been made, and it has been noted that r is an even function of time.)

Now let us assume that the Reynolds number is high enough that there is a range of k large compared to energy containing wavenumbers but small compared to v_o/ν, so that $\nu k^2 \ll v_o k$. For such k we may anticipate that the behavior of $r(k,\tau)$ should be dominated by the energy range mixing action discussed previously and consequently be characterized by a time the order of $(v_o k)^{-1}$. If this be so, the term $\nu k^2 r(k,\tau)$ on the left of (5.4) should be negligible compared to $\dot{r}(k,\tau)$, except in the immediate vicinity of the origin. Now, following arguments similar to those used in obtaining (5.2), let us retain on the right of (5.4) only those contributions involving b such that E(q) refers to the energy containing range

and only those contributions involving a such that E(p) or E(q) refer to this range. Making the further approximations used in obtaining (5.2), taking either $E(q)/E(k)$ or $E(p)/E(k) \approx 1$, as appropriate, using the fact that

$$a(k,p,q) \approx \sin^2\beta \qquad (q \ll k),$$

and writing

$$\int_o^\infty ds = \int_o^\tau ds' + \int_\tau^\infty ds, \text{ where } s' = \tau - s,$$

we obtain the asymptotic equation

$$\dot{r}(k,\tau) = -v_o^2 k^2 \left\{ \int_o^\tau g(k,\tau-s')r(k,s')ds' + \int_\tau^\infty [r(k,\tau-s)g(k,s) - g(k,s-\tau)r(k,s)]\, ds \right\}. \tag{5.5}$$

The solution satisfying (3.12) is $r(k,\tau) = g(k,|\tau|)$, since for this choice the second integral on the right of (5.5) vanishes and the equation reduces to the asymptotic form taken by (5.2) for the case $\nu k^2 \ll v_o k$. Thus we have the result

$$r(k,\tau) = J_1(2v_o k\tau)/(v_o k\tau). \tag{5.6}$$

A consistency check similar to that described after (5.3) shows that this solution is indeed asymptotically valid if k satisfies the double inequality we have postulated. The frequency spectrum corresponding to our solution is

$$\hat{r}(k,\omega) = \frac{2}{\pi}\left[1 - \left(\frac{\omega}{2v_o k}\right)^2\right]^{1/2} \qquad (\omega \leq 2v_o k), \tag{5.7}$$

$$= 0 \qquad (\omega > 2v_o k).$$

This function is shown in Figure 3.

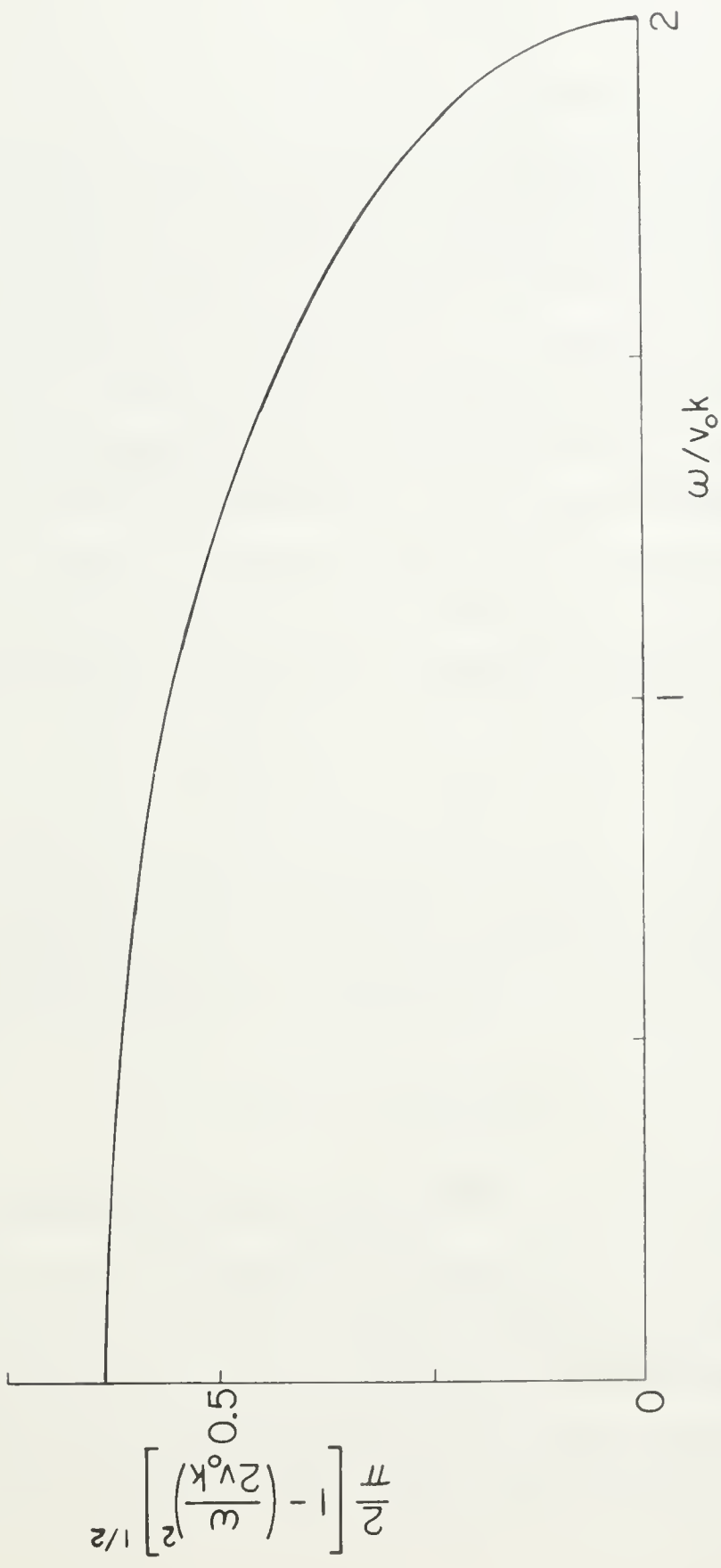


Figure 3. The frequency-spectrum function $(2/\pi)\left[1-(\omega/2v_o k)^2\right]^{1/2}$.

The spectrum of a correlation function must be positive-definite, and the fact that the $\hat{r}(k,\omega)$ given above has this property is some evidence of the consistency of our theory and of the realizability of solution-distributions obeying our statistical hypotheses. The sharp cut-off in the spectrum at $\omega = 2v_o k$ is a consequence of the asymptotic nature of our results. We have explicitly neglected the time variation of mode k due to interactions with modes of wavenumber appreciably greater than k. At any high but finite k, the interaction with higher wavenumbers, having higher characteristic frequencies, must contribute a slight tail to the spectrum, extending to infinite frequency. The height of the tail decreases with increasing separation of k from the energy containing region.

6. THE INERTIAL RANGE

Let us assume the existence of an extended range of k which satisfies all the inequalities invoked in Section 5 and for which the dissipation term on the left of (4.2) is negligible. The energy balance equation then is simply

$$\frac{1}{2}\iint_{\Delta} S(k|p,q)\,dpdq = 0. \tag{6.1}$$

It follows according to the analysis of Section 5 that the contributions to this equation for which p or q is in the energy containing range are given asymptotically by 2E(k) times the right side of (5.5) for $\tau = 0$. The s integrals in this expression vanish identically (independently of the form of r(k,s) and g(k,s)), which suggests that modes of high wavenumbers do not receive energy from direct interaction with the energy containing range. This is in accord with the widely

accepted intuitive notion that the transfer of energy up the spectrum proceeds by an essentially local cascade process. We shall assume that this is so, determine the spectrum in the inertial range, and then check the consistency of our assumption by obtaining an expression for the energy transport due to interaction among distant wavenumbers.

The existence of an effectively local cascade process, together with the conservative character of the interaction, implies that in the inertial range, where dissipation is negligible, the transport power must satisfy

$$\Pi(k) = \varepsilon \tag{6.2}$$

where the parameter ε is the mean power eventually dissipated at very high wavenumbers. In accord with the concept of a local cascade process we shall assume that contributions to $\Pi(k)$, as given by (4.7) and (4.3), may be neglected if p, q, or k' is very large or small compared to k. For all the remaining contributions, the relevant response functions and correlation functions are of the form $J_1(2v_o k\tau)/(v_o k\tau)$. The Θ factors defined by (4.4) then are symmetric in their arguments and have the dimensional form

$$[\Theta(k,p,q)] = [v_o]^{-1}[k]^{-1}$$

Consequently $\Pi(k)$ has the dimensional form

$$[\Pi(k)] = [v_o]^{-1}[k]^3[E(k)]^2$$

It seems rather clear from this that if $\Pi(k)$ is to be determined wholly by contributions with k', p, and q all in some essentially local neighborhood of k and is to have the value ε, which is independent of k, then one must have

$$E(k) = f(0)(\varepsilon v_o)^{1/2} k^{-3/2}, \tag{6.3}$$

where f(0) is a numerical constant. (We choose this symbol for later convenience.)

The argument just given only suggests the necessary form of E(k), and we must check to see that (6.3) actually satisfies (6.1) and leads to a local cascade process. If (6.3) is substituted in (4.3) and one takes $g(k,s) = g(v_o ks)$, $r(k,s) = r(v_o ks)$ (without specifying the particular functional forms of r and g), it is not difficult to verify by formal manipulations, using the identities (3.9), that (6.1) is satisfied. This actually is rather clearly implied by the conservative nature of the theory and the dimensional considerations above. The formal property has meaning, however, only if the integrals involved converge properly, which corresponds physically to the presumed localness of cascade. We may verify the latter property by considering the total power input to all modes of wavenumber $k'' > k'$ from direct interactions with all mode pairs p, q such that p or $q < k \ll k'$, where k and k' are fixed wavenumbers and all wavenumbers concerned are within the inertial range. Using $r(k,s) = g(k,|s|) = J_1(2v_o ks)/(v_o ks)$ and computing this power as

$$\Pi(k'|k) = \frac{1}{2}\int_{k'}^{\infty} dk'' \iint_{\Delta''}^{(p \text{ or } q < k)} S(k''|p,q)\,dp\,dq, \tag{6.4}$$

we find, asymptotically,

$$\Pi(k'|k) = (\text{numerical factor})\,[f(0)]^2 \varepsilon (k/k')^{3/2} \qquad (k \ll k'). \tag{6.5}$$

The triad interactions involved in the integration are shown in Figure 4. $\Pi(k'|k)$ goes to zero with k/k', and the cascade process is asymptotically local, as originally assumed. The dependence on k/k' is not particularly strong, however, and the cascade is rather ill-defined.

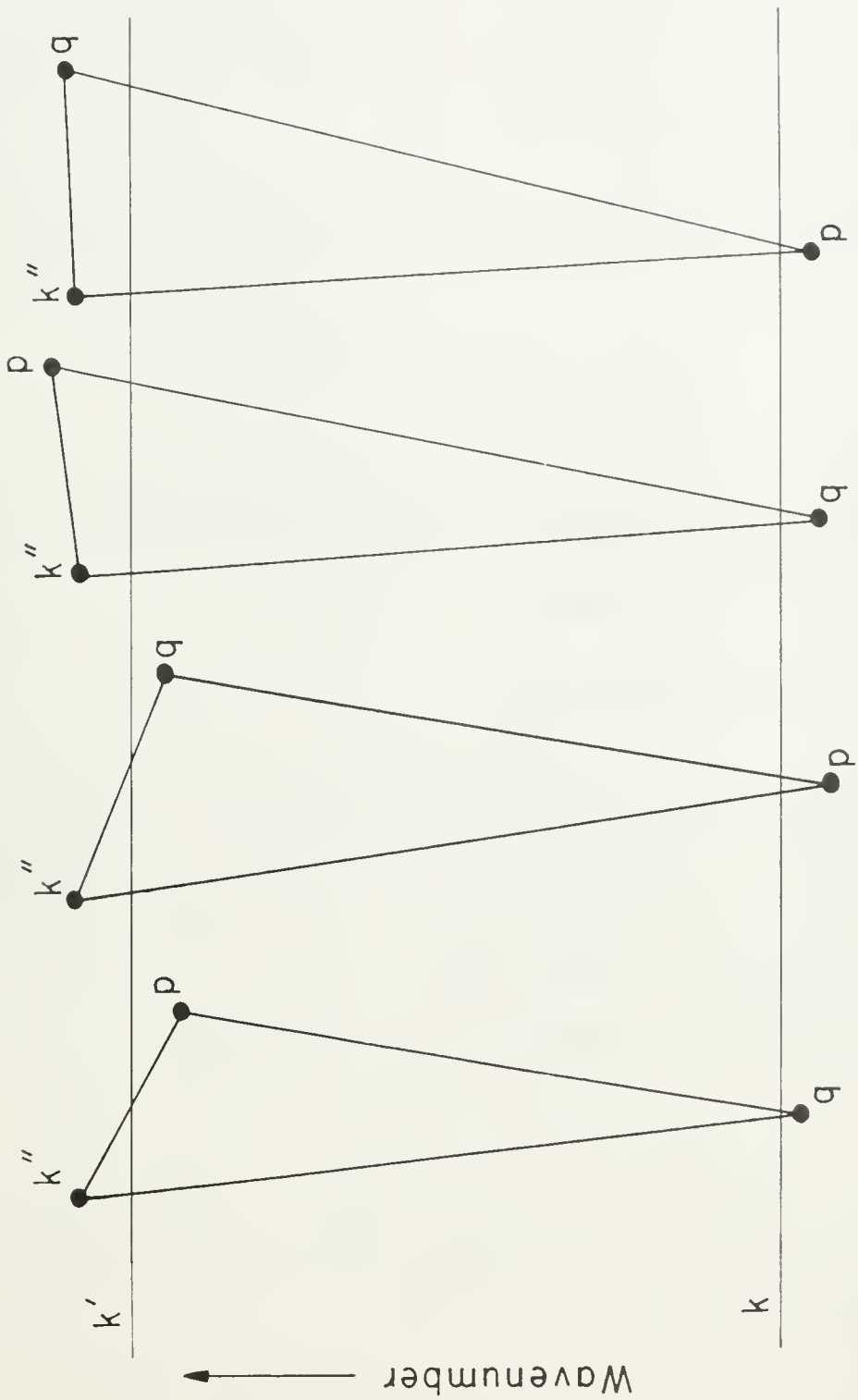


Figure 4. Triad interactions contributing to the distant-transport power $\Pi(k'|k)$.

It remains to determine the constant f(0). This may be done by substituting (6.3) into (4.7), taking the form $J_1(2v_o ks)/(v_o ks)$ for the response and correlation functions, evaluating the definite multiple integrals, and noting (6.2). After this is done, (5.6) and (6.3) provide a complete asymptotic solution for the inertial range.

If the inertial range extends through and well above the wavenumber k, the total energy above k according to (6.3) is

$$\int_k^\infty E(p)dp \approx \tfrac{1}{2} f(0)(\varepsilon v_o)^{1/2} k^{-1/2}. \tag{6.6}$$

Now let us consider the extrapolation of this expression down to low k. Assuming that f(0) is crudely the order of unity, it is clear that (6.6) gives an energy the order of v_o^2 (per unit mass) when k is the order of

$$k_o = \varepsilon/v_o^3. \tag{6.7}$$

Such an extrapolation violates our conditions for the inertial range, of course, but nevertheless this result suggests strongly that k_o is actually a wavenumber characteristic of the energy containing range. It is also known empirically that this is the case(Batchelor, 1953, p.103). We therefore may define a Reynolds number characteristic of the energy containing range by

$$R_o = v_o k_o^{-1}/\nu = v_o^4/\varepsilon\nu. \tag{6.8}$$

If the inertial range extends through and well below a wavenumber k, the total power dissipated below k is

$$(6.9) \qquad \int_0^k 2\nu p^2 E(p)dp \approx \frac{4}{3} f(0)\nu(\varepsilon v_0)^{1/2} k^{3/2}.$$

This expression becomes the order of ε for k the order of

$$(6.10) \qquad k_d = \varepsilon^{1/3} v_0^{-1/3} \nu^{-2/3} = R_0^{2/3} k_0.$$

It is clear, then, that necessary conditions for the validity of our asymptotic inertial range spectrum are

$$(6.11) \qquad k_0 \ll k \ll R_0^{2/3} k_0,$$

for an extended range of k. This requires in turn that R_0 satisfy

$$(6.12) \qquad R_0^{1/3} \gg 1.$$

Since $v_0/\nu = R_0 k_0$, it is clear that the condition $\nu k^2 \ll v_0 k$, invoked in the derivation of (5.6), is contained in (6.11).

A Reynolds number R_λ may be defined by $R_\lambda = v_0 k_\lambda^{-1}/\nu$, where $k_\lambda^2 = \varepsilon/15 v_0^2 \nu$ (Batchelor, 1953, p.51). Thus,

$$(6.13) \qquad R_\lambda = (15 R_0)^{1/2},$$

$$k_\lambda = R_\lambda k_0/15.$$

The condition (6.12) is equivalent to

$$(6.14) \qquad R_\lambda^{2/3} \gg 1.$$

7. THE DISSIPATION RANGE AT HUGE REYNOLDS NUMBERS

7.1. The integral equation for E(k)

The discussion at the end of Section 6 indicated that a major part of the dissipation takes place at wavenumbers the order of $k_d = R_o^{2/3} k_o$. This suggests that for $R_o^{1/3} \gg 1$, where the inequality is strong enough, all but a negligible fraction of the total dissipation occurs at wavenumbers $k \ll R_o k_o = v_o/\nu$. We shall assume that this is so and verify later that the resulting equations do yield a dissipation which falls off rapidly for $k \gg k_d$. Then for k in the dissipation as well as the inertial range, $r(k,s) \approx g(k,|s|)$ is given by (5.6). Let us anticipate, on the basis of the local cascade process already found for the inertial range, that energy transport in both inertial and dissipation ranges is negligibly dependent on direct contributions from triad interactions with modes outside both these ranges. Then the energy balance equation (4.2) takes the asymptotic form

$$(7.1) \qquad 2\nu k^2 E(k) = k \iint_{\Delta} [k^2 a(k,p,q) E(p) - p^2 b(k,p,q) E(k)] E(q) \theta_I(k,p,q) \frac{dpdq}{pq},$$

where

$$(7.2) \qquad \theta_I(k,p,q) = \int_o^\infty \frac{J_1(2v_o ks)}{v_o ks} \frac{J_1(2v_o ps)}{v_o ps} \frac{J_1(2v_o qs)}{v_o qs} \, ds.$$

As before, we have taken F(k) to be negligible for $k \gg k_o$. It is clear, physically, that the solution of (7.1) cannot be unique, since we have not specified the power input or dissipation rate. (In effect, we have pushed the source of energy supply off to zero wavenumber.) To specify a unique solution we may require

$$\int_0^\infty 2\nu k^2 E(k)\,dk = \varepsilon. \tag{7.3}$$

If we write

$$E(k) = (\varepsilon\nu_0)^{1/2} k^{-3/2} f(k/k_d), \tag{7.4a}$$

which may be expressed in the dimensionless form

$$E(k)/\nu_0\nu = (k/k_d)^{-3/2} f(k/k_d), \tag{7.4b}$$

we readily find that (7.1) reduces to the universal equation

$$2u^2 f(u) = \iint_\Delta \left[u^2 a(u,v,w)\left(\frac{u}{v}\right)^{3/2} f(v) - v^2 b(u,v,w) f(u)\right] w^{-3/2} f(w) K\left(\frac{v}{u},\frac{w}{u}\right)\frac{dv\,dw}{vw}\,, \tag{7.5}$$

where

$$K(m,n) = \int_0^\infty \frac{J_1(2s)}{s}\,\frac{J_1(2ms)}{ms}\,\frac{J_1(2ns)}{ns}\,ds. \tag{7.6}$$

Equation (7.3) takes the form

$$\int_0^\infty u^{1/2} f(u)\,du = \frac{1}{2}\,. \tag{7.7}$$

Instead of imposing (7.7), we could require that f(0) have the value determined in Section 6 from (6.2). The two conditions are equivalent because the nonlinear interaction is conservative; the energy transported through the inertial range is equal to the total viscous dissipation.

The singularities in the integrand of (7.5) at v = 0 and w = 0 cancel, as may be verified from (3.8) and (3.9). This, again, expresses the localness of the cascade process, according to which the total contribution to the integrand

from the infinitesimal neighborhood of these points is infinitesimal. We may anticipate that $f(u)$ is a well-behaved function in the entire range $0 \leq u \leq \infty$.

It was pointed out in Section 4 that the emission term in the energy balance equation is proportional to $E(k)$ so that, if the spectrum level in the immediate neighborhood of k were suddenly raised above its equilibrium value, the result would be a decrease of the net nonlinear input to the neighborhood and a return to equilibrium. This behavior suggests that it should be possible to solve (7.5) by a corresponding correction or iteration procedure in which one obtains an improved trial function as an appropriate linear combination of an initial trial function $2u^2 f(u)$ and the function generated as the right side of (7.5) upon substituting the initial function. It may be expected, however, that considerable practical difficulty will arise because of the singularities in the integrand.

It is of interest at this point to discuss further the relative roles played in (5.4) by local and by energy containing modes. We have noted that for $k \ll R_o k_o$ the behavior of $r(k,\tau)$ and $g(k,\tau)$ as functions of τ is determined by interactions involving the energy containing region. However, although contributions involving the energy containing region dominate the right side of (5.4) away from the origin, they cancel out in the immediate neighborhood of $\tau = 0$, as indicated by (5.5). In this neighborhood, $\dot{r}(k,\tau)$ goes to zero and the term $\nu k^2 r(k,\tau)$, which here dominates the left side, is balanced by contributions on the right nearly entirely from local triads. At $\tau = 0$, this corresponds to the localness of energy transport. As the inequalities $k_o \ll k \ll R_o k_o$ become stronger, the range of τ about the origin in which wholly local contributions dominate becomes a smaller fraction of $(v_o k)^{-1}$.

7.2. Behavior for $k \gg k_d$

Let us suppose that R_o is sufficiently large that there are values of k, very large compared to k_d, for which (7.1) is still valid. There appear to be two general possibilities for the energy supply to such wavenumbers. Either they are powered by an essentially local cascade process, or they draw power principally from wavenumbers, the order of k_d, in the region where the spectrum begins to fall below the inertial range form. We might perhaps anticipate that the first possibility is the actual fact. In the inertial range we have seen that the energy transport is essentially local. Let us now imagine a part of the spectrum to be depressed by local viscous drain. It would appear that the net input from much lower wavenumbers should be affected relatively little, since to start with the local region was already very far from equipartition with these wavenumbers. On the other hand, the local decrease in the level of excitation with respect to that of closer wavenumbers should act in the direction of increasing relatively more the net input from these latter wavenumbers. These considerations actually must be considerably elaborated because of the nonlinearity of the system, but it is not difficult to verify that the transport does remain essentially local in the dissipation range and to find the qualitative shape of the spectrum for $k \gg k_d$.

First, it is easy to see that no law of the form $E(k) \propto k^n$ is possible for $k \gg k_d$. Consider, to start, the assumption that the transport is essentially local so that the right side of (7.1) is dominated by contributions for which q and p are the order of k. Then in the contributing regions $\Theta_I(k,p,q)$ is of order $(v_o k)^{-1}$. Since a and b are dimensionless, one finds, counting up the wavenumber factors, that (7.1) corresponds to the order-of-magnitude relation

(7.8) $$2\nu k^2 E(k) \sim v_o^{-1} k^2 [E(k)]^2 .$$

As k increases indefinitely this cannot be satisfied by any power law except the absurd choice n = 0. On the other hand, if we assume that most of the power is supplied by interactions involving modes of wavenumber $\sim k_d$, then in place of (7.8) we have

(7.9) $$2\nu k^2 E(k) \sim v_o^{-1} k_d k E(k_d) E(k),$$

where leading terms have been retained on the right. The expression $k_d k$ appears instead of k^2 because one of the pair p, q is of order k_d and the integration ranges of both p and q are restricted to intervals of width $\sim k_d$. Equation (7.9) cannot be satisfied by any power law at all. Thus, both (7.8) and (7.9) are inconsistent with the initial assumption of a power law, under which they were derived.

The impossibility of a power law arises from the fact that E appears linearly on the left of (7.1) and bilinearly on the right, while the equation is homogeneous in the wavenumbers. This suggests that E(k) contains a factor e^{-ck/k_d} (where c is a numerical factor), since that form gives the product law $e^{-cp/k_d} e^{-cq/k_d} = e^{-c(p+q)/k_d}$. An investigation of (7.1) for $k \gg k_d$ shows that it is, in fact, satisfied asymptotically by a function of the form

(7.10) $$E(k) = \text{const}\ (k/k_d)^2 e^{-ck/k_d} \qquad (k \gg k_d).$$

The dominant contributions to the power input come from wavenumber pairs p, q forming very flat triangles with k, as would be anticipated from the effect of

the exponential factors in depressing the value of $E(p)E(q)$ for pairs not having this property. The integration thus is effectively confined to an area of the p, q plane very much smaller than k^2, and in this area $E(p)E(q) \gg [E(k)]^2$. The factor $(k/k_d)^2$, which might seem surprising in view of the homogeneity of (7.1) in wavenumbers, arises because of the restriction of integration domain; its form also is affected by the behavior of $a(k,p,q)$, which vanishes for an exactly flat triangle. As we partially anticipated on general grounds, the energy transport is local; this appears as a consequence of the fact that the fraction of p, q pairs satisfying $p + q \approx k$, and such that p or $q \sim k_d$, goes to zero with k_d/k.

The form of (7.10) confirms the assumption that the dissipation is effectively confined to wavenumbers the order of k_d. It suggests also that each decade range, say, of wavenumbers $k \gg k_d$ dissipates most of the energy supplied to it, passing on an ever-decreasing fraction to the next decade. Corresponding to this fact, the emission terms in (7.1) are very small compared to the absorption term for $k \gg k_d$.

An important characteristic of (7.10) is that it suggests the existence of the moments $\int_o^\infty k^n E(k)dk$ for arbitrarily high n. In the x space representation, this implies the existence of mean square velocity derivatives of all orders - a property which seems required by physical intuition. To establish this conclusion more fully it is necessary to investigate the behavior of $E(k)$ for extremely high wavenumbers $k \gg R_o k_o$. Although we have not determined the form of $r(k,\tau)$ in this range, the energy balance equation is nevertheless tractable because the function $\theta(k,p,q)$ appearing in the dominant absorption term is determined asymptotically by $g(k,s) \approx \exp(-\nu k^2 s)$. An analysis of the type described above shows that

(7.11) $$E(k) \propto k^3 \exp[-(\text{const})k] \qquad (k \gg R_o k_o).$$

This result appears to be valid even for $R_o^{1/3} \sim 1$. The proportionality constants, however, are not independent of R_o.

7.3. Skewness and flatness factors of the distribution of $\partial^n \tilde{u}_1 / \partial x_1^n$

For $R_o^{1/3} \gg 1$, the skewness factor

(7.12) $$S_1 = \langle (\partial \tilde{u}_1 / \partial x_1)^3 \rangle / \langle (\partial \tilde{u}_1 / \partial x_1)^2 \rangle^{3/2}$$

is given in terms of spectrum moments by the asymptotically valid relation (Batchelor, 1953, p. 168)

(7.13) $$S_1 = -\frac{3}{7}(30)^{1/2} \nu \left[\int_o^\infty k^2 E(k) dk \right]^{-3/2} \int_o^\infty k^4 E(k) dk.$$

By (7.4) and (7.7), we find

(7.14) $$S_1 = -\frac{12}{7}(15)^{1/2} R_o^{-1/6} \int_o^\infty u^{5/2} f(u) du.$$

We can also evaluate S_1 by expressing it in terms of k space triple moments and expressing these moments in terms of $U(k,\tau)$ and $g(k,\tau)$ by the method employed in Paper I to obtain (3.7). This same procedure can be applied in general to the determination of the skewness factor S_n of the distribution of $\partial^n \tilde{u}_1 / \partial x_1^n$. It is not difficult to show that for $R_o^{1/3} \gg 1$ the result depends asymptotically only on the dissipation range structure and that upon substituting (5.3), (5.6), and (7.4) in the resulting expressions we obtain the asymptotically valid result

(7.15) $$S_n = A_n R_o^{-1/6},$$

where the A_n are universal constants.

The R_o dependence indicated by (7.14) and (7.15) is quite weak, and it is impossible to estimate, without further analysis, how strong the inequality $R_o^{1/3} \gg 1$ has to be for the asymptotic behavior not to be masked by other effects.

The application of the theory to fourth-order moments, which is outlined in Paper I, yields an expression in terms of $U(k,\tau)$ and $g(k,\tau)$ for the flatness factor of the distribution of $\partial^n u_1/\partial x_1^n$. It can be seen from this expression that the flatness factor approaches for each n some universal value asymptotically independent of R_o, for large $R_o^{1/3}$.

7.4. A visualization of the dynamics in terms of linear systems

An interesting paradox arises when we attempt to interprete the dynamics of huge Reynolds number turbulence, as given by the present theory, in terms of familiar linear systems. Equation (6.7) implies that, if the driving forces were removed, the characteristic decay time for the energy containing range of the turbulence, due to the power output to higher wavenumbers, would be the order of $(v_o k_o)^{-1}$; that is, the energy range is approximately critically damped by a resistive dynamical coupling to higher wavenumbers.

Now let us go up the wavenumber spectrum considering each decade, say, a sub-system. As we go through the inertial range, the characteristic time of each sub-system is of order $(v_o k)^{-1}$, the energy in each sub-system is of order $(\varepsilon v_o)^{1/2} k^{-1/2}$, and the power output to the next higher sub-system is $\sim \varepsilon$. The "damping factor" of the sub-system, the fraction of energy passed on in a characteristic "period", is of order

$$\varepsilon (v_o k)^{-1} / (\varepsilon v_o)^{1/2} k^{-1/2} = (k_o/k)^{1/2}.$$

Thus, the damping due to dynamical coupling goes steadily down as we rise through the inertial range.

When we get into the dissipation range $k \gtrsim k_d$, a similar argument shows that the damping due to coupling with the next higher sub-system continues to go down, and with great rapidity since the coupling to higher modes becomes ineffective and transfers a rapidly decreasing fraction of the energy in a sub-system. In this range, as in the inertial range, the damping factor associated with direct viscous dissipation increases linearly with k (as $\nu k^2/v_o k$); however, it does not become critical until $k \sim R_o k_o$, which is far above the range of significant dissipation for huge Reynolds numbers. Thus we have a situation in which the energy containing range, where viscosity does not act, is approximately critically damped, while the range in which viscosity disposes of the energy is very lightly damped! As we go up above $R_o k_o$, the damping due to dynamical coupling continues to decrease rapidly (it has long since become extremely small), while the direct viscous damping becomes, finally, very large and dominates the mechanics of the sub-systems.

The behavior described seems strange, but there does not appear to be anything either physically or mathematically inconsistent about it. It arises from the nonlinearity and complication of the system, and the consequent artificiality of dividing it into sub-systems. The characteristic frequencies of the sub-systems are not entirely internal parameters, but are strongly affected by the energy range mixing, as discussed before. We must also keep in mind that each sub-system serves as a kind of frequency multiplier for the energy supplied to it.

The asymptotic identity of time correlation and response functions for the inertial and dissipation ranges indicated by the present theory is of

considerable fundamental interest. It has been shown for many types of conservative systems that time correlation and averaged response functions are identical in an equilibrium (detailed balance) ensemble. In particular, this is true for a dissipationless system closely related to turbulence (Paper I, appendix). In the present case, we have found this same behavior for modes which are very far from a state of detailed balance and which dissipate most of the energy lost by the system. It seems likely that this is connected with the fact that for these modes the effective damping due to both dynamical coupling and direct dissipation is very small.

8. COMPARISON WITH EXPERIMENT

8.1. The effect of deviations from asymptotic conditions

The analysis of the present paper has been based on exact isotropy and stationarity, and we have indicated solutions of the equations only for extremely high Reynolds numbers. It is not possible to make a very meaningful comparison with experiment without some idea of the theoretical corrections implied by the unavoidable experimental deviations from asymptotic conditions. Unfortunately, not very much can be said on this matter at the present stage. At the least, a semiquantitative estimate of the simplest deviations must wait upon evaluation of the function $f(k/k_d)$ which characterizes the asymptotic solution itself. For the present we can attempt only a very qualitative discussion.

First, we may consider the relation between our fictional stationary isotropic turbulence and freely decaying isotropic turbulence, which possibly can be closely approximated in the laboratory. For R_o sufficiently high that the asymptotic solution developed in Section 7 has a range of accurate validity,

it seems clear that the behavior of this range should be negligibly affected by decay, provided enough time has elapsed for the high wavenumber structure to have been fully established. This follows from the fact that the decay of v_o is very slow compared to characteristic periods of the range. The spectrum and time correlation for the stationary case therefore should apply to the free decay case, provided v_o and ε are measured within a time short compared to the decay time. Since the instantaneous rate of viscous dissipation equals the instantaneous rate of decrease of kinetic energy, it should not matter whether ε is measured as $-\partial(\frac{3}{2}v_o^2)/\partial t$ or as $\int_o^\infty 2\nu k^2 E(k)dk$, if accurately isotropic and homogeneous conditions have been established.

At lower Reynolds numbers the equivalence of the stationary and decay cases no longer follows, and we must consider also that (7.1) no longer is necessarily an accurate expression of the stationary theory. After the numerical constants are evaluated, (6.5) may be of value in the qualitative understanding of lower Reynolds numbers.

The role of anisotropy in the present theory is quite interesting. Let us consider huge Reynolds number homogeneous axisymmetric turbulence. For this case the response and time correlation functions for high wavevectors can be determined in close analogy to the isotropic case, yielding similar Bessel function expressions. It is easy to see, however, that the effective mixing velocity, replacing v_o in the arguments, depends on the angle between $\underset{\sim}{k}$ and the axis of symmetry and is maximum when $\underset{\sim}{k}$ is in the direction of maximum rms velocity. The energy transport among a triad $\underset{\sim}{k}$, $\underset{\sim}{p}$, $\underset{\sim}{q}$ in the inertial or dissipation ranges has a consequent dependence on the orientations of these wavevectors relative to the axis of symmetry. Thus, despite the localness of energy cascade, it does

not appear likely that the present theory yields asymptotic isotropy of energy partition in the inertial and dissipation ranges. The geometry of the transport terms is sufficiently complicated that further investigation is needed to determine whether the anisotropy in the θ factors actually results in appreciable energy anisotropy and, if so, whether it is of the same or opposite sign as the anisotropy of the rms velocity components.

As we go to wavenumbers $k \gtrsim R_o k_o$, the transport terms begin to be dominated by isotropic factors of the form $\exp(-\nu k^2 s)$ in the response functions. In consequence of the localness of the cascade process, we should then expect an approach to energy isotropy at these high wavenumbers, regardless of possible anisotropy lower in the spectrum. This should be true regardless of the size of R_o.

The presence of inhomogeneity as well as anisotropy brings into question the basic hypotheses of the present theory. If, for example, we consider turbulence confined within a region not large compared to k_o^{-1}, the mode density will not be high, and we cannot expect the weak dependence property to hold accurately. However, the weak dependence theory may be expected to give an exact description of turbulence homogeneous in only one or two directions - for example, flow in an infinitely long pipe or channel. To obtain closed, determinate covariance equations, weak dependence need be assumed only for wavevectors differing in their axial components; the phase correlations in transverse directions may be retained and determined, along with the mean velocity profile.

Without further investigation, extreme caution must be used in attempting to interpret shear flow experiments on the basis of homogeneous theory. Laboratory "spectrum" measurements involve a kind of "local Fourier analysis" and give results which depend on the position in the inhomogeneous

flow. The Fourier components used in the present theory are not local quantities; they refer to the entire flow. Without further justification, we cannot divide into local regions an inhomogeneous flow, in which there are necessarily phase correlations between pairs even of high-lying Fourier modes, and attempt to treat each region more or less independently. It is conceivable that the mixing action of a strong shear velocity in one part of the flow may influence the energy balance in high wavenumber Fourier components which affect a laboratory "spectrum" measurement made in a different part of the flow.

Apart from the interpretation of the present theory, the concept of local Fourier analysis is not a simple one. Terms like "eddies of wavenumber k", sometimes used in discussing turbulence, imply a vagueness of distinction between x space and k space concepts which can lead to serious inconsistencies.

We shall now proceed to disregard the cautions just presented and compare the asymptotic stationary, isotropic theory with real experiments, including shear flow experiments. It is hoped that the comparison may lead to insights regarding the questions which have been raised. A further motivation is that the only other course open at present would appear to be to ignore experiments entirely.

8.2. Turbulent pipe flow

The highest laboratory values of R_o ($> 10^3$) appear to have been obtained in shear flows. An experimental situation for which very careful measurements have been taken is the fully developed flow in a long circular pipe (Laufer, 1954). Measurements were made, at several stations along the pipe radius, of the one-dimensional spectra $\phi_1(k_1)$, $\phi_2(k_1)$, $\phi_3(k_1)$ of the energy in the axial, radial, and circumferential velocity components, respectively.

The results of the measurements on the axis are shown in Figure 5. The one-dimensional spectrum of the total kinetic energy $\phi(k_1) = \phi_1(k_1) + \phi_2(k_1) + \phi_3(k_1)$, for two stations, is shown in Figure 6. In most cases the points shown on the latter plot are not experimental points but are computed from the curves drawn by Laufer through the original experimental points for the three component-spectra. In both Figures 5 and 6, wire length corrections have been applied only to $\phi_1(k_1)$. The values of $R_o^{1/3}$, k_o, and k_d shown on Figure 6 are computed from $v_o^2 = \frac{1}{3}\langle \tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2 \rangle$ and estimates of local values of ε. The dissipation rate cannot be determined precisely from the measurements, but the computed parameters are not very sensitive to its value. $R_o^{1/3}$ and k_d probably are not in error by more than 10% and k_o probably by not more than 25%. The accuracy of the relative values of the component spectra may be estimated as about 15 - 20% (Laufer, private communication).

It can be seen from Figure 6 that the total energy spectrum follows a $k_1^{-3/2}$ law quite closely for a substantial fraction of the range between k_o and k_d. If estimated wire-length corrections (Laufer, private communication) are applied to $\phi_2(k_1)$ and $\phi_3(k_1)$, the range of adherence to the $k_1^{-3/2}$ law is somewhat extended. Within the range where $\phi(k_1)$ exhibits -3/2 law behavior, it can be seen from Figure 5 that there are substantial differences among the slopes of the component spectra, the deficiency in excitation of u_2 and u_3 at low wavenumbers changing to an excess, above isotropic relation to u_1, at high wavenumbers.

Superficially at least, the behavior would seem not inconsistent with what might be expected on the basis of the asymptotic isotropic results. The total energy cascade appears to proceed according to the isotropic law. As the cascade goes on, there appears to be a drain of energy from u_1 to u_2 and u_3 so that the component spectra have respectively greater and lesser slopes than the

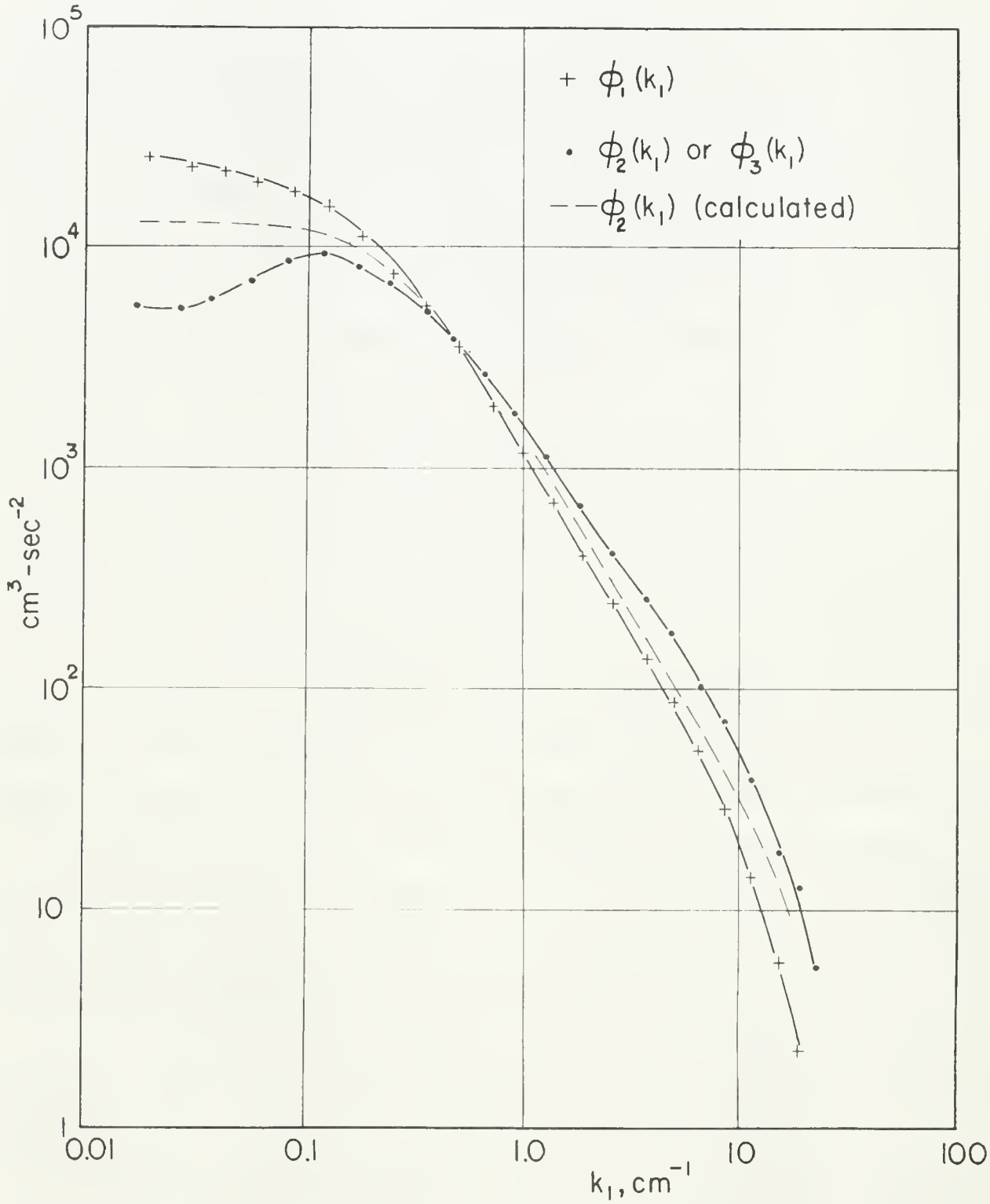


Figure 5. Energy-component spectra $\phi_1(k_1)$, $\phi_2(k_1)$, $\phi_3(k_1)$ on the axis of fully turbulent pipe flow. The dashed line gives $\phi_2(k_1)$ as calculated from $\phi_1(k_1)$ using isotropic relations. The pipe radius is a = 12.33 cm, and the Reynolds number based on a and the mean velocity on axis is 5×10^5. After Laufer (1954).

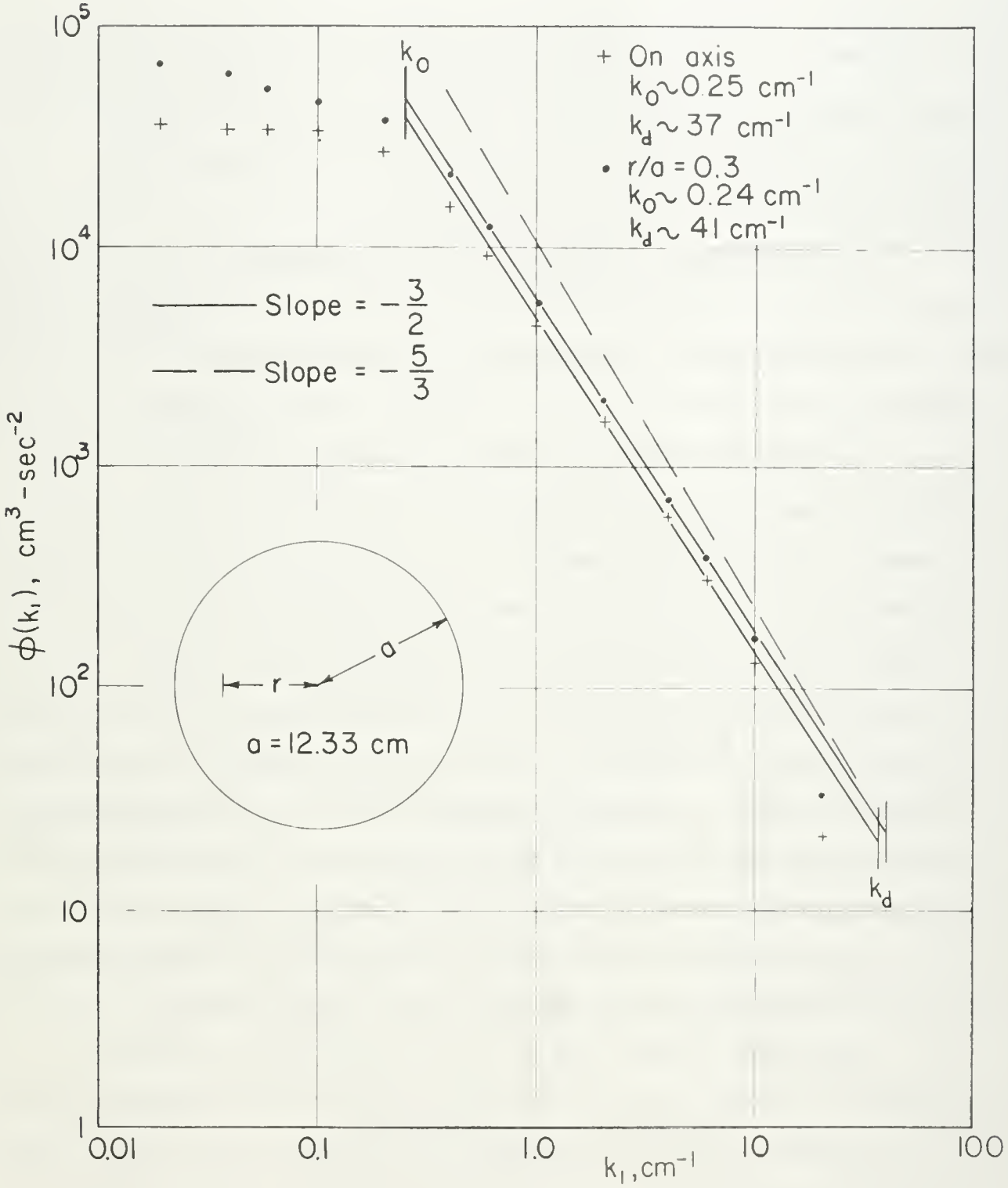


Figure 6. Total energy spectrum $\phi(k_1) = \phi_1(k_1) + \phi_2(k_1) + \phi_3(k_1)$ at two stations in the flow of Figure 5. $R_o^{1/3} \sim 15$, on axis, ~ 17 at r/a = 0.3. From data of Laufer (1954).

total spectrum. The eventual nonisotropic partition of energy at high wavenumbers does not appear too surprising in view of the discussion in Section 8.1. We may note that $R_o k_o$, the wavenumber at which the response functions must begin to become isotropic, is far above the range plotted.

In the original interpretation of the measurements (Laufer, 1954), it was noted that $\phi_1(k_1)$ follows a $k_1^{-5/3}$ law very closely over a substantial range. This is the inertial range law predicted by the Kolmogorov theory for isotropic turbulence, and the result was taken as indicating consistency with that theory. The comparison does not seem very satisfying, however, as Laufer noted. In the center of the inertial range, $\phi_1(k_1)$ represents only about 20 - 25% of the total energy associated with k_1, and the other components, which contain most of the energy, do not follow the -5/3 law at all. In making a comparison with an isotropic theory it would appear more reasonable to consider the total energy spectrum, as we have done above. For this spectrum, the -3/2 law appears to give an optimum straight-line fit, and the -5/3 law appears to lie outside the experimental error. It may also be noted that the very fact of the observed anisotropy at very high wavenumbers, which is not surprising on the basis of the present theory, seems a complete mystery on the Kolmogorov theory.

It should be emphasized that the discussion above is merely suggestive. The cautions expressed in Section 8.1. must be kept firmly in mind.

Measurements in the boundary layer on a flat plate at values of $R_o^{1/3}$ similar to those in the pipe flow experiment have been reported by Klebanoff (1955). In the outer part of the boundary layer a behavior qualitatively similar to that described above was found; the slope of $\phi_1(k_1)$ was steeper, and that of $\phi_2(k_1)$ flatter, than -3/2. Unfortunately, measurements of $\phi_3(k_1)$ were not made.

It is of interest that the anisotropy at very high wavenumbers found in the pipe flow measurements did not appear in this experiment.

Betchov (1957) has also reported measurements at closely comparable $R_o^{1/3}$, made in a shear flow of novel character. Only $\phi_1(k_1)$ was measured, and it resembled closely the corresponding function for the pipe flow. No meaningful comparison with isotropic theory seems possible in the absence of data on $\phi_2(k_1)$ and $\phi_3(k_1)$.

8.3. Grid turbulence

Grid produced turbulence appears to represent at present the closest laboratory approach to homogeneity and isotropy. Although Reynolds numbers high enough to produce an inertial range have not been described in the literature, it is of some interest to examine to what extent the dissipation range spectrum at moderate Reynolds numbers obeys the scaling indicated by (7.4b). Figure 7 shows the results of one-dimensional spectrum measurements by Stewart and Townsend (1951) as plotted by these authors on the basis of scaling indicated by the Kolmogorov theory. Similar measurements have been reported by Liepmann, Laufer, and Liepmann (1951). Figure 8 shows the curves drawn by Stewart and Townsend through their data points, rescaled according to (7.4b). The plots have been prepared so that the curves for $R_M = 5250$ are congruent in each case. The values of R_M correspond to the values shown in the table below:

R_M	R_o	$R_o^{1/3} = R_o k_o/k_d$	$R_o^{-2/3} = k_o/k_d$
2625	13	2.35	0.18
5250	26	2.95	0.11
10500	52	3.75	0.07

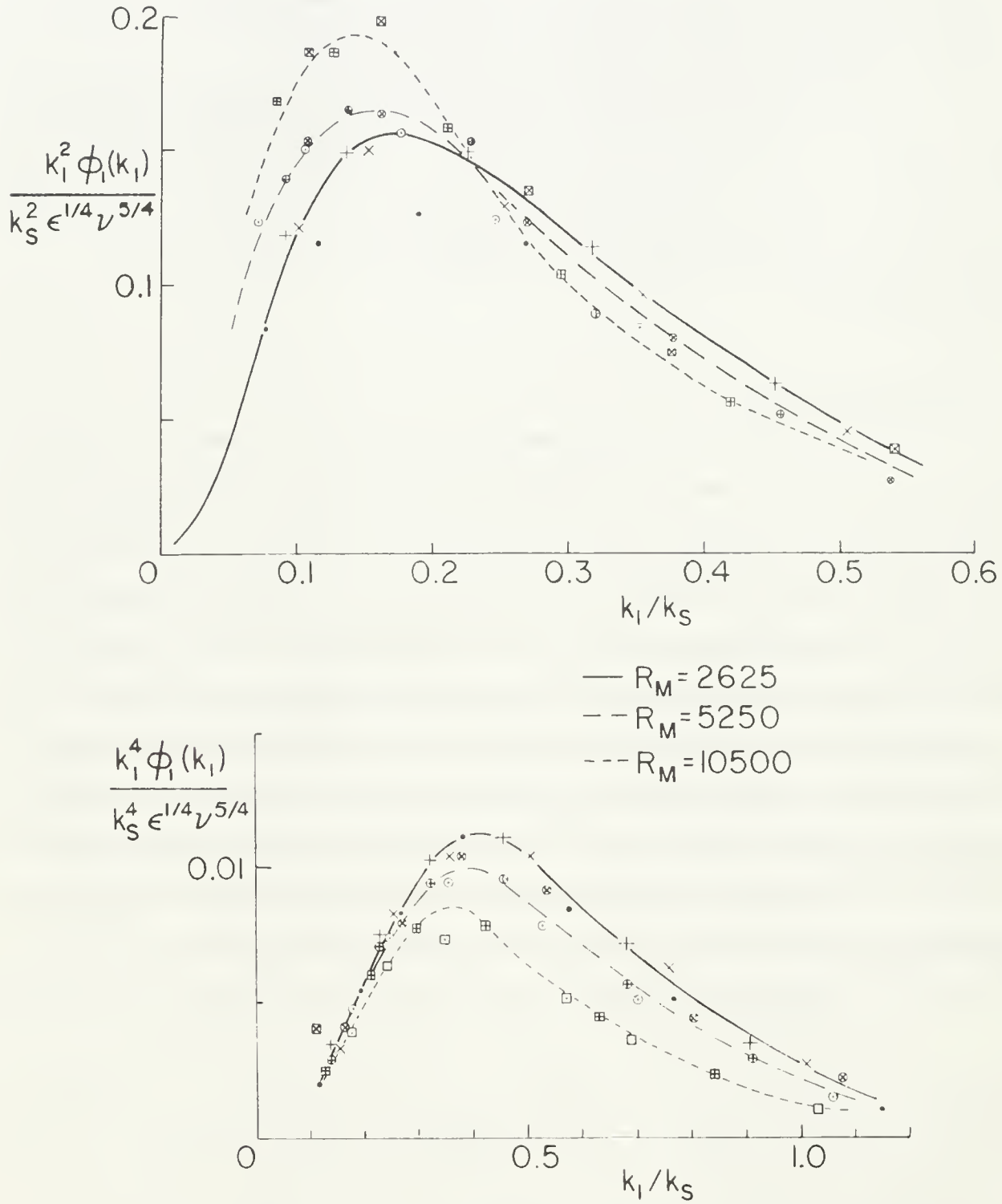


Figure 7. Dimensionless plots of $k_1^2 \phi_1(k_1)$ and $k_1^4 \phi_1(k_1)$ according to Kolmogorov scaling ($k_s = R_o^{3/4} k_o = \epsilon^{1/4} \nu^{-3/4}$), for several grid Reynolds numbers R_M. After Stewart and Townsend (1951).

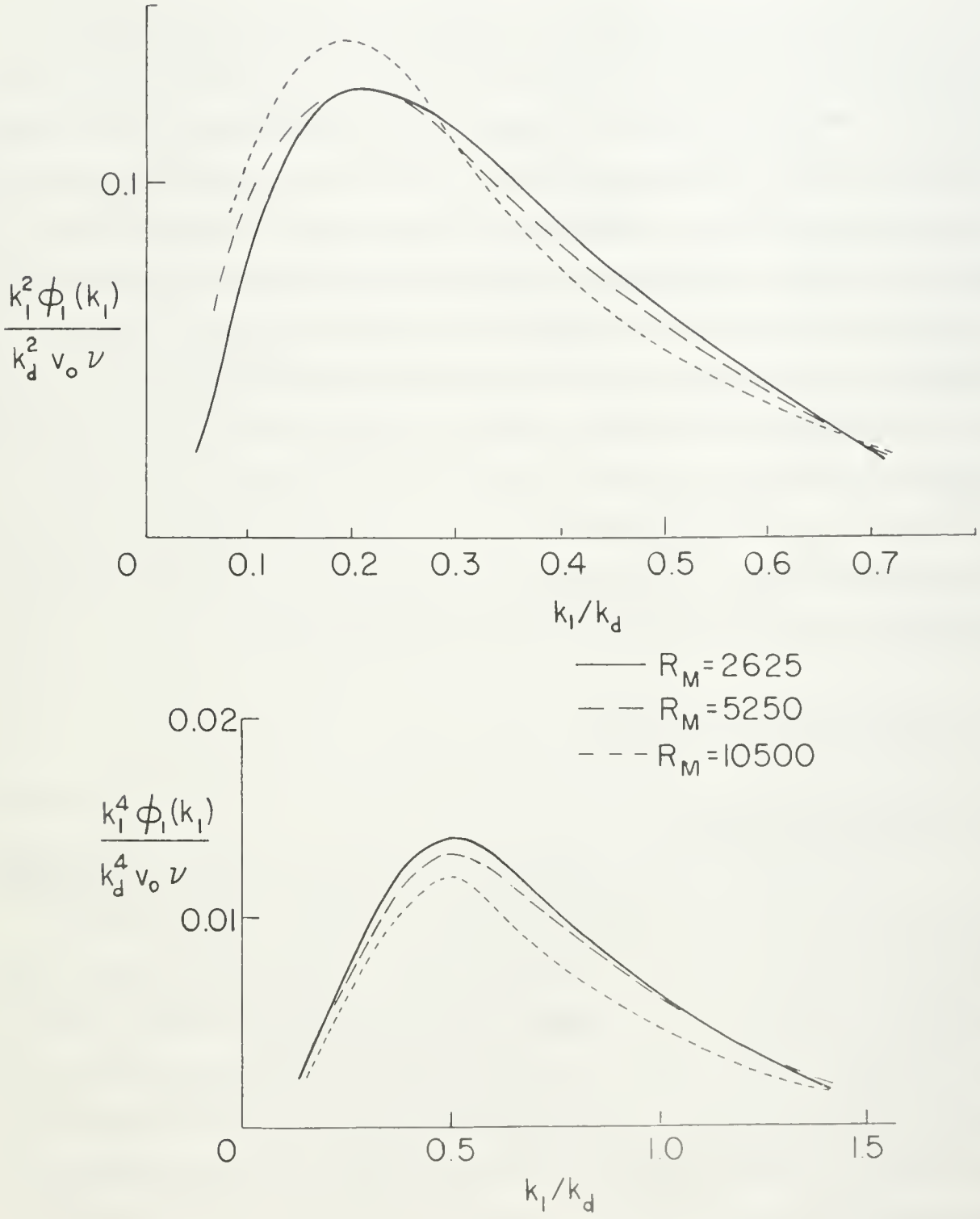


Figure 8. The curves in Figure 7 rescaled according to the present theory ($k_d = R_o^{-1/12} k_s$, $v_o \nu = R_o^{1/4} \varepsilon^{1/4} \nu^{5/4}$).

It will be noticed that the curves of both $k_1^2\phi_1(k_1)$ and $k_1^4\phi_1(k_1)$ fall substantially closer together with the new scaling than with the old. In particular, the systematic shift with R_M of the horizontal position of the maxima seems largely absent with the new scaling. It should be noted that, while $R_o k_o$ has a definite theoretical significance as the wavenumber at which $\nu k^2 = v_o k$, k_o is only nominally the wavenumber characterizing the energy containing range. In this experiment most of the energy lies below k_o in each case. From the values given in the table it would seem not implausible that, particularly in the case of $k_1^4\phi_1(k_1)$, the functions should bear some relation to the asymptotic forms for a substantial part of the k range plotted. However, the values of $R_o^{1/3}$ are too low, and the experimental scatter too high, to permit attaching very much significance to the comparison. It must also be noted that detailed measurements on the isotropy of the flow were not presented.

9. COMPARISON WITH OTHER THEORIES

9.1. The Kolmogorov theory

The Kolmogorov theory (Kolmogorov, 1941; Onsager, 1945; v. Weizsäcker, 1948; reviewed by Batchelor, 1953, chap. 6) has occupied a central place in thinking on turbulence in the past decade because of the intuitive appeal of its assumptions, the economy of its methods, and the approximate empirical support for its predictions. Certain predictions of the present weak dependence theory appear to differ only slightly from those of the Kolmogorov theory: the spectrum law in the inertial range by $(k/k_o)^{1/6}$ and the characteristic scaling wavenumber in the dissipation range by $R_o^{-1/12}$. A detailed comparison of the two theories is not possible because one is wholly quantitative and the other consists of dimensional arguments. However, we shall see that their postulatory bases are fundamentally in conflict.

Kolmogorov's original assertion was essentially that the internal dynamics of the sufficiently fine-scale structure in x space, at high Reynolds number, should be independent of the large scale motion; the latter should, in effect, merely convect local regions bodily. The restatement of this hypothesis by v. Weizsäcker holds that the dynamical interaction of the Fourier modes $\underline{u}(\underline{k})$ is effectively local in wavenumber space and that low k modes exert on sufficiently high k modes only an action effectively equivalent to convection by a suitably random uniform velocity field. This implies that the transport of energy from low to high wavenumbers proceeds by a cascade process and that the internal mechanism of the cascade process at high k is quite independent of the original source of the energy. Then it can be seen that the spectrum in the inertial range depends only on ε, whence, by dimensional analysis, $E(k) \propto \varepsilon^{2/3} k^{-5/3}$. In contrast, our inertial range spectrum (6.3) involves the energy range parameter v_o.

It is not hard to see that the weak dependence hypothesis introduced in Section 2 is inconsistent with the idea that the low k motion has the convective effect of a random uniform velocity field on the high k dynamics. As an illustration, consider an ensemble of one-dimensional flows consisting of a Gaussianly distributed uniform velocity v, with mean square v_o^2, and a superposed zero-mean fluctuation field $\tilde{u}(x) = \sum_k u(k) e^{ikx}$, $u(-k) = u^*(k)$. For simplicity, let us suppose that the u(k) are sufficiently weak that nonlinear interaction among them is negligible. Let us assume that at t = 0 the various u(k,0) are statistically independent of each other and of the uniform field v. Then for t > 0 we have

$$u(k,t) = u(k,0) e^{-ivkt}, \tag{9.1}$$

which expresses the convective action of v. If we now imagine that at time t' some external agency produced a small impulsive disturbance in u(k), this disturbance would simply remain and be convected also. Therefore the impulse response function for u(k) is simply

$$\zeta(k;t,t') = e^{-ivk(t-t')} \qquad (t \geq t'). \tag{9.2}$$

The normalized time correlation function and averaged impulse response function for u(k) are then

$$\frac{\langle u(k,t)u^*(k,t+\tau)\rangle}{\langle |u(k,0|^2\rangle} = r(k,\tau) = \langle e^{ivk\tau}\rangle = \exp(-\tfrac{1}{2}v_o^2k^2\tau^2), \tag{9.3}$$

$$\langle \zeta(k;t+\tau,t)\rangle = g(k,\tau) = \exp(-\tfrac{1}{2}v_o^2k^2\tau^2) \qquad (\tau \geq 0), \tag{9.4}$$

where we have used the assumption that the v distribution is Gaussian.*

Now let us consider the moment

$$M = \frac{\langle \zeta(k;t+s,t)u(p,t)u(q,t)u^*(p,t+s)u^*(q,t+s)\rangle}{\langle |u(p,0)|^2\rangle\langle |u(q,0)|^2\rangle}$$

By direct evaluation from (9.1) and (9.2) we have

$$M = \exp\left[-\tfrac{1}{2}v_o^2(k-q-p)^2s^2\right] \qquad (s \geq 0), \tag{9.5}$$

where the independence of the u(k,0) has been used. If, instead, the weak dependence hypothesis were asserted, we would have from (9.3) and (9.4)

---

* Note that the ensemble averages here are not equivalent to neighborhood averages. This represents a fundamental difficulty in the explicit formulation of the Kolmogorov-v. Weizsäcker concepts.

$$(9.6) \qquad M \approx g(k,s)r(p,s)r(q,s) = \exp[-\tfrac{1}{2}v_o^2(k^2+p^2+q^2)s^2] \qquad (s \geq 0).$$

For k-p-q = 0, which corresponds to the triangle relation of wavevectors in our three dimensional theory, the weak dependence hypothesis would give a very wrong answer indeed. It is clear that, in the case taken, statistical independence of simultaneous quantities does not imply weak dependence of non-simultaneous quantities.

The analysis just given indicates why, in actual three-dimensional turbulence, the weak dependence theory gives a dependence of high wavenumber dynamics on v_o, in conflict with the Kolmogorov hypothesis. The Θ factors in the energy balance equation (7.1) are derived from moments of the type just considered. For $\underset{\sim}{p} + \underset{\sim}{q} = \underset{\sim}{k}$ they could not have a dependence on v_o, as indicated by (7.2), if the energy containing modes effectively exerted on high wavenumbers a simple convective action as implied by the Kolmogorov theory. It is of some interest, in this connection, arbitrarily to replace the Θ factors in (7.1) by expressions of the form

$$(9.7) \qquad \Theta_I'(k,p,s) = \int_o^\infty r(\omega_k s)r(\omega_p s)r(\omega_q s)ds,$$

where

$$\omega_k = k[kE(k)]^{1/2}$$

is a "local" characteristic mode frequency as suggested by the Kolmogorov theory. With this change, (7.1) leads to an inertial range law of form $E(k) \propto \varepsilon^{2/3}k^{-5/3}$, and we find $\omega_k \propto \varepsilon^{1/3}k^{2/3}$.

The action of low k modes on high k modes implied by the weak dependence theory can be described as follows. Let us suppose that at t = 0 the various $\underset{\sim}{u}(\underset{\sim}{k})$ are weakly interdependent, and let us then switch on the dynamical interaction. We notice from (3.1) that there is an essential difference between the actions of modes of low, non-zero wavenumber and modes of zero wavenumber (uniform field). A mode $\underset{\sim}{k}' = 0$ couples mode $\underset{\sim}{k}$ to itself, while a mode $\underset{\sim}{k}' \neq 0$ (but however small) couples mode $\underset{\sim}{k}$ to another mode with which it has only a weak statistical relation. Now let us take three high wavenumber modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, $\underset{\sim}{q}$ and consider the effect upon them of the presence of a large number of excited modes of non-zero wavenumbers $\ll k, p, q$. The three modes are not coupled each to itself, yielding coherent time dependencies of the type (9.1). Instead, each mode has mixed into it energy from many closely adjacent modes with which it has only weak statistical relations and which, in turn, are only weakly related to the modes being mixed into the other two modes. In this situation, the cancellation of time factors expressed by (9.5) does not occur. It can be shown that many low k modes must contribute to the mixing action if the picture is to be self-consistent. This, of course, is ensured by our limit $L \to \infty$.

It should be noted that the presence of a true uniform (zero-wavenumber) velocity field does not invalidate the weak dependence hypothesis. A uniform field v simply adds to all the correlation and response functions a factor $e^{i\underset{\sim}{v}\cdot\underset{\sim}{k}}$ or $e^{-i\underset{\sim}{v}\cdot\underset{\sim}{k}}$, as is clear from (3.1). Averaging over an ensemble of true uniform fields is inadmissible.

The difference of action between uniform fields and low k fields may be described in x space as follows. Suppose our large box of dimension L is filled with initially weakly dependent high k excitation. If a low k motion exists, it will shear and distort each initial harmonic component; this is just

the mixing action described above. The shearing distortion appears locally to be very small, but it is clear from (3.1), and the discussion above, that the mixing of adjacent high k modes proceeds at a rate which depends on the magnitude of the velocity and not of the shear associated with the low k motion. The locally tiny shearing actions add up to an important cumulative effect. This points up the fact, noted earlier, that it is necessary to use great care in combining concepts from the x space and k space representations. Quite apart from whether it is profitable to think of turbulence as consisting of local regions in x space, the use of Fourier components gives a representation of the problem which is intrinsically nonlocal in x space. Each $\underset{\sim}{u}(\underset{\sim}{k})$ refers to the entire box.

Turning from questions of representation to physics, we should note at this point that the local properties of fine-scale dynamics appealed to in the original Kolmogorov hypothesis hardly seem to be implied by the qualitative empirical behavior of turbulent fluids. The fine structure at high Reynolds numbers does not consist of a uniform distribution of little random blobs but rather of a substantially intermittent distribution of sharply defined, extended, and tangled vortex filaments and sheets (Batchelor, 1953, Section 8.4). The transfer of energy from low to high wavenumbers appears to involve, in x space, the stretching of these filaments and sheets. This behavior is readily observed by stirring ink into a bathtub of water ($R_o > 10^4$ is easily achieved). A typical filament appears to extend throughout a substantial part of the total turbulent domain.

The spatial structure described does not seem consistent with the idea of fine scale isotropy within local regions asserted by Kolmogorov (1941). Isotropy, when it exists, appears to be a statistical property shown by averages over large space domains containing many local regions. The fact that the space correlation may be isotropic for small separations does not at all imply isotropy within local regions. The confusion of these two properties leads to unwarranted

dynamical assumptions and has constituted a fundamental illogic in the use of the Kolmogorov ideas. In general, isotropy for small correlation distances implies isotropy of local averages in k space (see Section 2), but only because the $\underset{\sim}{u}(\underset{\sim}{k})$ give a highly nonlocal representation of the x space structure.

It is of interest to note, finally, that the preservation of incompressibility by the pressure forces during the stretching of filaments implies analytically certain long-range ordering effects in the fluid (Batchelor and Proudman, 1956).

9.2. Heisenberg's heuristic theory

Several authors have proposed models for the transport power $\Pi(k)$ based on simple physical analogies (Batchelor, 1953, chap. 6). Of these, the heuristic theory of Heisenberg (1948) has proved the most popular. Heisenberg assumed that $\Pi(k)$ could be represented as a power dissipation by all wavenumbers $< k$ due to the action of an effective "eddy viscosity" which represented their coupling to all wavenumbers $> k$. In analogy to the expression for dissipation by real viscosity, he took

$$\Pi(k) = \int_0^k 2\left\{\kappa \int_k^\infty [q^{-3}E(q)]^{1/2} dq\right\} p^2 E(p)\,dp, \tag{9.8}$$

where κ is a disposable numerical constant. The eddy viscosity $\{\}$ was obtained by dimensional arguments on the basis of the Kolmogorov-v. Weizsäcker hypothesis of the localness of k space dynamics. Because of its dimensional structure, (9.8) leads to a $k^{-5/3}$ inertial range spectrum law. At very high k it leads to a k^{-7} law, implying the non-existence of mean-square third- and higher-order velocity derivatives. The possible relation of (9.8) to (2.2) has been investigated by Tchen (1954).

The comparison of the eddy viscosity and weak dependence theories is made more illuminating by first modifying (9.8) to yield a $k^{-3/2}$ inertial range law; Heisenberg's basic physical assumption seems independent of the conflict between the weak dependence and Kolmogorov theories. Accordingly we shall consider the transport function

$$\Pi(k) = \int_o^k 2\left\{\kappa' v_o^{-1} \int_k^\infty q^{-1} E(q)dq\right\} p^2 E(p)dp, \tag{9.9}$$

where κ' is a new disposable numerical constant. The eddy viscosity $\{\ \}$ is here obtained by dimensional considerations under the requirement that it depend inversely on v_o, as implied by our previous considerations. The rationality of (9.9) makes it perhaps more satisfying intuitively than (9.8).

From (9.9) we immediately obtain

$$-\frac{\partial \Pi(k)}{\partial k} = 2\kappa' k (v_o k)^{-1}\left[\int_o^k \frac{p^3}{k} E(k)E(p)\,\frac{dp}{p} - \int_k^\infty k^2 E(k)E(p)\,\frac{dp}{p}\right]. \tag{9.10}$$

This equation has been written in a form to facilitate comparison with $-\partial \Pi(k)/\partial k$ as given by the right side of (7.1). The time factor $(v_o k)^{-1}$ corresponds to the θ_I factor in (7.1). The most immediate qualitative difference between the two expressions is that (9.10) expresses the total interaction as an integral over pair interactions (k,p) rather than the triad interactions (k,p,q) which enter (7.1), and which are fundamental to the structure of the original equation of motion (3.1). A further difference concerns the sharp distinction in (9.10) between the interactions with modes greater than or less than k. The absorption term involves only $p < k$ and the emission term only $p > k$. In (7.1), both sorts of modes enter both terms, and the net input appears as the balance of both

ingoing and outgoing contributions from every triad interaction; this more closely resembles other dissipative equilibrium situations in statistical physics.

E(k) appears as a factor in both the absorption and emission terms of (9.10), suggesting that this transport expression does not give the tendency to return to equilibrium, upon a perturbation of the spectrum in the neighborhood of k, that was noted in the discussion of (7.1). This peculiarity shows up more clearly in (9.9), and also in the unmodified form (9.8). If the equilibrium is disturbed by an initial increase of spectrum level in the range $> k$, then by (9.8) or (9.9) the effective eddy viscosity acting on wavenumbers $< k$ is increased. This will lead to an increased flow of energy from wavenumbers $< k$ to those $> k$ which will tend to increase further the eddy viscosity acting on the lower wavenumbers. A possible instability is indicated. On general statistical mechanical grounds, we should expect precisely contrary behavior; an increase of excitation in modes $> k$ should lead to a decreased flow of energy to them from modes $< k$, as on the weak dependence theory.

It can be seen from (9.10) that for k very large compared to the wavenumbers which contribute appreciably to the dissipation integral $\varepsilon = 2\nu\int_0^\infty p^2 E(p)dp$, the absorption term is dominated by contributions $p \ll k$ in the principal (real) viscous dissipation region. Thus, the transport does not proceed by local cascade for such k, as it does in the weak dependence theory.

Most of the comparisons made above can also be made on the basis of (9.8), but with less clarity. For this case, the irrationality of the eddy viscosity expression results in greater asymmetry between absorption and emission terms.

9.3. The analytical theories of Heisenberg and Chandrasekhar

The hypothesis that fourth-order moments of the distribution of the $\underset{\sim}{u}(\underset{\sim}{k})$ are related to second-order moments as in an independent distribution implies normal relation of the corresponding x space moments. It is a much stronger statement than, and contains, the hypothesis of weak dependence for these moments. From this it is clear that the independent relation hypothesis is also inconsistent with the Kolmogorov theory. As discussed in Section 2, the independent relation hypothesis is inadmissible on both empirical and theoretical grounds. It nevertheless is of interest to compare the analytical theories of Heisenberg (1948, Section 5) and Chandrasekhar (1955), which are founded upon it, with the present theory.

Heisenberg's analytical theory properly is appropriate to freely decaying turbulence, but its essential structure may be discussed in terms of our present stationary specialization of formalism. (For discussion of the inertial and dissipation ranges at high Reynolds number, the stationary and decay cases are equivalent.) The theory involves first expressing the triple moment $S(k,\tau)$, defined by (3.5), in terms of fourth-order moments by substituting for $u_m(\underset{\sim}{q},t)$ the bilinear expression obtained by integrating (2.2) with respect to time. Next, the fourth-order moments are expressed in terms of $U(k,\tau)$ by means of the independent relation hypothesis. This results in a closed equation for $U(k,\tau)$. The analysis was carried out for $\nu = 0$, but can be straightforwardly generalized to $\nu \neq 0$.

The expression found for $S(k,\tau)$ can be obtained from our result (3.7) by 1) replacing the response functions $g(p,s)$, $g(q,s)$ with $\exp(-\nu p^2 s)$, $\exp(-\nu q^2 s)$; 2) discarding the terms involving $a(k,p,q)$; 3) retaining from the terms involving

$b(k,p,q)$ only a particular linear combination of parts which are antisymmetric in k and p for $\tau = 0$. The alteration of the response functions is equivalent to ignoring the effect on them of the nonlinear interaction. The discarding of terms amounts, on the basis of the weak dependence theory, to throwing out most of the interaction altogether. This shows up most strikingly if one evaluates the contribution to $r(k,\tau)$, for high k, due to energy range mixing, after the fashion in which (5.5) and (5.6) were obtained. One finds that this contribution vanishes as $k/k_o \to \infty$. This result apparently led Heisenberg to a curious chain of reasoning. Independence of $r(k,\tau)$ on v_o clearly is absurd in a stationary coordinate system, but it is just the behavior one might expect, on the basis of the Kolmogorov - v. Weizsäcker considerations, in a sort of quasi-Lagrangian coordinate system moving with the low k motion. Heisenberg concluded, therefore, that possibly the independent relation hypothesis, and consequently his statistical equation of motion (Heisenberg, 1948, equation (83)), might really be appropriate to such a coordinate system. He proceeded, on this basis, to seek solutions $r(k,\tau)$ of a form similar to that used in (9.7). There seems to be no analytical support whatever for this.

The theory of Chandrasekhar (1955) was developed by analysis in x space. Expressed in k space, it differs from that of Heisenberg, as generalized above, only in that $S(k,\tau)$ is expressed in terms of fourth-order moments by substituting a bilinear integral expression for $u_i^*(\underset{\sim}{k},t-\tau)$ rather than for $u_m(\underset{\sim}{q},t)$. This gives, however, quite a different result from Heisenberg's, thereby indicating the intrinsic inconsistency of the independent relation hypothesis. Chandrasekhar's result may be obtained from (3.7) by replacing $g(k,s)$ with $\exp(-\nu k^2 s)$ and discarding all the contributions involving $b(k,p,q)$.

The terms retained are all rigorously positive-definite for $\tau = 0$, and the theory consequently does not yield conservation of energy by the interaction (Kraichnan, 1957) as does Heisenberg's prescription. However, the theory has the mathematical virtue of being expressible by a partial differential rather than an integro-differential equation in x space.

The inconsistency of the independent relation hypothesis with the Kolmogorov theory is clearly illustrated by the attempt of Chandrasekhar (1956) to reconcile his theory with the $k^{-5/3}$ inertial range law. In order to obtain the formal possibility of a space correlation function consistent with this law, it was found necessary to assume that the full space-time correlation function for small space-time separations was independent of v_o. This conflicts seriously with the Kolmogorov theory, which, as we noted, requires such behavior for the correlation function in a quasi-Lagrangian, not a Eulerian coordinate system. It also conflicts with the basic physics of the problem.

Our discussion has indicated that in both Heisenberg's and Chandrasekhar's theories the use of the independent relation hypothesis throws away substantial parts of the statistical interdependence and dynamical coupling of the Fourier modes. It should be noted that essentially equally significant omissions could be expected of possible theories made determinate by assuming that certain sixth, eighth, or higher order cumulants vanish instead of fourth order. As the order increases, the number of the neglected individual cross-moment cumulants entering physically significant expressions rises rapidly, and their total contribution cannot be expected to decrease with increase of order. This situation is connected with the nonlinearity of (3.1), through which phase relations of large groups of modes are established by networks of triad

interactions. The weak dependence theory retains cumulants of all orders. The presence of g in the expressions for third, fourth, and higher order moments (Paper I) indicates the effect, on individual cross moments, of the inter-relation of all the modes.

9.4. The theories of Proudman and Reid, and Tatsumi

Proudman and Reid (1954) and Tatsumi (1957) assert the independent relation hypothesis only for the distribution of simultaneous values of the $\underset{\sim}{u}(\underset{\sim}{k},t)$. They are able, thereby, to obtain closed equations for the time change of E(k) in freely decaying isotropic turbulence. No predictions are obtained concerning time correlations. The analytical structure of these theories is complicated, and we shall only indicate very briefly the relation with the present theory for the inertial range at very high Reynolds numbers. It should be noted that these theories were primarily intended to describe the energy containing range.

We shall assume that there exists an inertial range, far above the energy containing range, in which energy transport takes place by local cascade and in which the direct effects of viscosity are negligible. In this range we shall assume that the spectrum obeys a k^{-n} law. Then it is not difficult to verify that the hypothesis of Proudman and Reid, and Tatsumi leads to an asymptotic expression for $-\partial \Pi(k)/\partial k$ at time t which differs from the right side of (7.1) only in that $\theta_I(k,p,q)$ is replaced by

$$\tau_o = \int_t^\infty \frac{E(k,s)}{E(k,t)}\, ds,$$

where E(k,t) is the value of E(k) at time t.

From empirical knowledge we know that τ_o must be of order $(v_o k_o)^{-1}$, the overall decay time of the turbulence, whereas we have seen that θ_I is of order $(v_o k)^{-1}$ for significant triad interactions. If arguments are followed similar to those which led to and verified (6.3), we now find, instead,

$$E(k) \sim (\varepsilon v_o k_o)^{1/2} k^{-2} \tag{9.11}$$

for the asymptotic inertial range spectrum. This result involves a contradiction: the integral of $2\nu k^2 E(k)$ over the inertial range now diverges toward low wavenumbers, a fact which is inconsistent with the initial assumption that viscosity effects were negligible. This suggests that the theories of Proudman and Reid, and Tatsumi do not yield an inertial range. A more detailed investigation confirms this conclusion.

The result found is not too surprising. The independent relation hypothesis is known empirically to be invalid for high wavenumbers (Batchelor, 1953, Section 8.2). The appearance of τ_o instead of θ_I in the transport expression would seem to indicate that the effect of ignoring the existence of fourth-order cumulants is to discard most of the effect of the dynamical interaction on the time correlation and response functions which enter θ.

ACKNOWLEDGEMENTS

The author would like to thank the many persons whose valuable comments, discussion, and criticism contributed to this paper. He is grateful to Dr. John Laufer for information concerning experimental errors and wire-length corrections, and to several authors for permission to reproduce Figures from their published papers.

REFERENCES

BATCHELOR, G. K. 1953 The Theory of Homogeneous Turbulence. Cambridge University Press.

BATCHELOR, G. K. and PROUDMAN, I. 1956 Phil. Trans. A, 248, 369.

BETCHOV, R. 1957 J. Fluid Mech. 3, 205.

CHANDRASEKHAR, S. 1955 Proc. Roy. Soc. A, 229, 1.

CHANDRASEKHAR, S. 1956 Phys. Rev. 102, 941.

HEISENBERG, W. 1948 Z. Physik, 124, 628.

KLEBANOFF, P. S. 1955 Nat. Adv. Comm. Aero., Wash., Rep. no. 1247

KOLMOGOROV, A. N. 1941 Doklady Akad. Nauk, S.S.S.R., 30, 301; 32, 16.

KRAICHNAN, R. H. 1957 Phys. Rev. 107, 1485.

KRAICHNAN, R. H. 1958 Phys. Rev. 109, No. 5 (in press) [Cited as Paper I].

LAUFER, J. 1954 Nat. Adv. Comm. Aero., Wash., Rep. no. 1174.

LIEPMANN, H. W., LAUFER, J., and LIEPMANN, K. 1951 Nat. Adv. Comm. Aero., Wash., Tech. Note no. 2473.

MILLIONSHTCHIKOV, M. 1941 Doklady Akad. Nauk, S.S.S.R., 32, 615, 619.

ONSAGER, L. 1945 Phys. Rev. 68, 286.

PROUDMAN, I. and REID, W. H. 1954 Phil. Trans. A, 247, 163.

STEWART, R. W. and TOWNSEND, A. A. 1951 Phil. Trans. A, 243, 359.

TATSUMI, T. 1957 Proc. Roy. Soc. A, 239, 16.

TCHEN, C.-M. 1954 Phys. Rev. 93, 4.

v. WEIZSÄCKER, C. F. 1948 Z. Physik, 124, 614.